9.5 Taylor series (泰勒級數) and Maclaurin series (馬克勞林級數)

Concept: If $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\cdots$.

What is $a_n, n=0,1,2,\cdots$?

Define: Taylor polynomial

If $f^{(n+1)}(x)$ is continuous on (a,b) and $c\in(a,b)$, ➔ $\exists z$ between x and c, ∍

$$f(x)=P_n(x)+R_n(x),$$

where $P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$ is called the nth Taylor polynomial for f at c,

$R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ is called the Lagrange form of the remainder and

Error $=|f(x)-P_n(x)|=|R_n(x)|\le\frac{M}{(n+1)!}|x-c|^{n+1}$, if $|f^{(n+1)}(x)|\le M, \forall x\in(a,b)$.

Define: Taylor series and Maclaurin series

If $f^{(n)}(x), \forall n=1,2,\cdots$ are continuous on (a,b) and $c\in(a,b)$, ➔

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$
$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor series for $f(x)$ at c. Moreover, if $c=0$, then the series

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

is called the Maclaurin series for $f(x)$.

Theorem: Convergence of Taylor series

If $f^{(n)}(x), \forall n = 1, 2, \cdots$ are continuous on (a,b) and $c \in (a,b)$.

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ holds $\Leftrightarrow$ $\exists\, z$ between x and c, such that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = 0, \forall x \in (a,b)$$

Ex 1: Find the Maclaurin series for $f(x) = e^x$.

Ex 2: Find the Maclaurin series for $f(x) = \sin x$.

Ex 3: Find the Taylor series for $f(x) = \ln x$ at $x = 1$.

Formula:

(1)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \forall |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \forall |x| < 1$$

(2)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall x \in \mathbb{R}$$

(3)

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}, \forall 0 < x \le 2$$

(4)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \forall x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \forall x \in \mathbb{R}$$

(5)

$$(1+x)^k = \begin{cases} 1 + C_1^k x + C_2^k x^2 + \cdots + C_k^k x^k, \text{if } k \in \mathbb{N} \\ 1 + C_1^k x + C_2^k x^2 + \cdots + C_n^k x^n + \cdots, \text{if } k \notin \mathbb{N} \text{ and } |x| < 1 \text{ if } k < 0, |x| \le 1 \text{ if } k > 0 \end{cases}$$

where $C_n^k = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$.

Ex 4: Find the power series for $f(x) = \sqrt[3]{1+x}$.

Ex 5: (1) Find the power series for $f(x) = \cos\sqrt{x}$.

(2) Use the result of (1) to compute $f^{(98)}(0)$.

Ex 6: Use a power series to approximate $\int_0^1 e^{-x^2}dx$ with an error of less than 0.01