9.3 Testing convergence

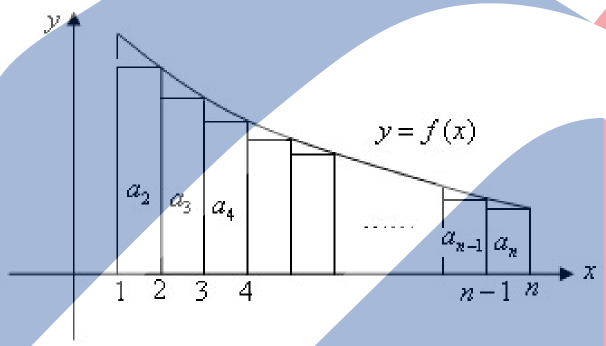
Theorem: The integral test

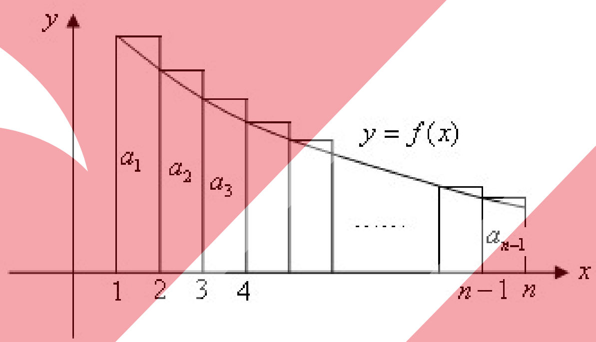
If $a_n = f(n), n = 1, 2, \cdots$, f is continuous on $[1, \infty)$ and $f(x) \searrow 0, \forall x \in [1, \infty)$,

then

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Leftrightarrow \int_1^{\infty} f(x)dx \text{ converges.}$$

Proof: By the figures, we have





$$\sum_{i=2}^{n} a_i = \sum_{i=2}^{n} f(i) \le \int_1^n f(x)dx \le \sum_{i=1}^{n-1} f(i) = \sum_{i=1}^{n-1} a_i$$

(➔) If $\sum_{i=1}^{\infty} a_i$ converges, then $\int_1^{\infty} f(x)dx$ converges.

(🡐) If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{i=2}^{\infty} a_i$ converges. So $\sum_{i=1}^{\infty} a_i$ converges.

Ex 1: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

Theorem: p-series

(1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow$ $p > 1$

(2) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges $\Leftrightarrow$ $p > 1$

Theorem: The comparison test

Let $0 < a_n \le b_n, \forall n = 1, 2, \cdots$, we have

(1) If $\sum b_n$ converges, then $\sum a_n$ converges.

(2) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Ex 2: Determine the series convergence or divergence.

(1) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(2) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2 + 1}$

Theorem: Limit comparison test

Suppose that $a_n > 0, b_n > 0, \forall n = 1, 2, \cdots$ and $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, we have

(1) If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(2) If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(3) If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ex 3: Determine whether the series converge or diverge.

(1) $\sum_{n=3}^{\infty} \frac{\sqrt{n}+8}{n^2-2n-1}$

(2) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{1000}}$

(3) $\sum_{n=1}^{\infty} \sin^2(\frac{1}{n})$

Define: The alternating series

If $a_n > 0, \forall n = 1, 2, \cdots$, then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$ is an alternating series.

Theorem: Alternating series test (Leibniz's rule)

Let $a_n > 0, \forall n = 1, 2, \cdots$, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if the following two conditions are satisfied:

(1) $a_n \geq a_{n+1}, \forall n \geq N$, for some integer N.

(2) $\lim_{n\to\infty} a_n = 0$.

Also, if the series has sum S, (i.e. $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$), then

$$|S - S_k| < a_{k+1}.$$

Where $S_k = \sum_{n=1}^{k}(-1)^{n-1}a_n$ is the kth partial sum of the series.

Ex 4: Does the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ converge?

Ex 5: Does the series $\sum_{k=1}^{\infty}(-1)^k\frac{\tan^{-1}k}{k}$ converge?

Ex 6: Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ correct to three decimal places.

Define: Absolutely and conditional convergences

(1) $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

(2) $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Ex 7: Show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is conditionally convergent.

Theorem: The Ratio test

Let $a_n \neq 0, \forall n = 1, 2, \cdots$ and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

(1) If $L < 1$, then $\sum a_n$ converges absolutely.

(2) If $L > 1$, then $\sum a_n$ diverges.

(3) If $L = 1$. The Ratio test is inconclusive.

Ex 8: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Ex 9: Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{e^n}$.

Theorem: The Root test

Let $a_n \neq 0, \forall n = 1, 2, \cdots$ and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

(1) If $L < 1$, then $\sum a_n$ converges absolutely.

(2) If $L > 1$, then $\sum a_n$ diverges.

(3) If $L = 1$. The Root test is inconclusive.

Ex 10: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.