9.2 Series

Define: For the infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$, the nth partial sum is given by

$$S_n = a_1 + a_2 + \cdots + a_n .$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges and

$$\sum_{n=1}^{\infty} a_n = S .$$

If the sequence $\{S_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Ex 1: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Ex 2: $\sum_{n=0}^{\infty} (-1)^n$

Theorem: Convergence of a Geometric series

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, 0 \le |r| < 1 \\ \text{diverges}, |r| \ge 1 \end{cases}$$

Ex 3: Find $\sum_{n=0}^{\infty} \frac{3}{2^n}$

Ex 4: Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

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Theorem: The divergence test

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Thus, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex 5: Determine whether the series $\sum_{n=0}^{\infty} \frac{n!}{2(n!)+3}$ is convergent or divergent.

Note: $\lim_{n\to\infty} a_n = 0$ is just a necessary condition, not a sufficient condition for series to be convergent.

Ex 6: Harmonic series (調和級數) : $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Theorem: Properties of infinite series

(1) If $\sum a_n$ and $\sum b_n$ are convergent series ➔ so is $\sum(\alpha a_n + \beta b_n), \forall \alpha, \beta \in \mathbb{R}$ and $\sum(\alpha a_n + \beta b_n) = \alpha \sum a_n + \beta \sum b_n$.

(2) If either $\sum a_n$ or $\sum b_n$ diverges and the other converges, ➔ $\sum(a_n + b_n)$ diverges.

(3) If $\sum_{n=1}^{\infty} a_n$ converges ➔ $\sum_{n=k}^{\infty} a_n$ is also converges.

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