9 Infinite series

9.1 Sequences (數列)

Define: Sequence

Assume that $a_n = f(n), n = 1, 2, \ldots,$ then $a_1, a_2, \ldots, a_n, \ldots$ or $\{a_n\}_{n=1}^{\infty}$ is called an infinite sequence (or sequence).

The number a_n is the nth term of the sequence.

Define: Convergent sequence

If $\lim_{n\to\infty} a_n = L$ (exists) ➔ the sequence $\{a_n\}$ converges to L, otherwise, the sequence diverges.

Theorem:

If $a_n = f(n), n = 1, 2, \ldots,$ and $\lim_{x\to\infty} f(x) = L(\text{or } \pm\infty)$, ➔ $\lim_{n\to\infty} a_n = L(\text{or } \pm\infty)$.

Ex 1: Find the limit of the sequence whose nth term is $a_n = (1+\frac{1}{n})^n$.

Ex 2: Discuss the convergence of the sequence:

(1) $\{(-1)^n\}$

(2) $\{2^n\}$

(3) $\{a_n = \dfrac{n^2}{2^n - 1}\}$

(4) $\{a_n = \dfrac{n!}{n^n}\}$

Define: Monotonic sequences and bounded sequences

A sequence $\{a_n\}$ is said to be

(1) increasing if $a_n \le a_{n+1}, \forall n \in \mathbb{N}$ or $a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots$ ($\nearrow$)

(2) decreasing if $a_n \ge a_{n+1}, \forall n \in \mathbb{N}$ or $a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots$ ($\searrow$)

(3) monotonic sequence if it is either increasing or decreasing.

(4) bounded above by M if $a_n \le M, \forall n \in \mathbb{N}$;

bounded below by m if $a_n \ge m, \forall n \in \mathbb{N}$;

bounded if $\exists m, M$, $\ni m \le a_n \le M, \forall n \in \mathbb{N}$

Theorem: Bounded monotonic sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Ex 3: Show that the sequence $a_n = \dfrac{1\cdot 3\cdot 5\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdots\cdot(2n)}$ converges.