8.2 Improper Integrals (瑕積分)

Case 1: Improper integrals with infinite limits of integration.

(1) If f is continuous on $[a,\infty)$ and

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \text{ exists} \Rightarrow \int_a^{\infty} f(x)dx \text{ converges.}$$

(2) If f is continuous on $(-\infty,b]$ and

$$\int_{-\infty}^{b} f(x)dx = \lim_{s\to-\infty}\int_s^b f(x)dx \text{ exists,} \Rightarrow \int_{-\infty}^{b} f(x)dx \text{ converges.}$$

(3) If f is continuous on $(-\infty,\infty)$ and $\forall c\in(-\infty,\infty)$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{s\to-\infty}\int_s^c f(x)dx + \lim_{t\to\infty}\int_c^t f(x)dx \text{ both limits exist,} \Rightarrow \int_{-\infty}^{\infty} f(x)dx \text{ converges.}$$

If the improper integral is not convergence (收斂) ➔ the improper integral is called divergence (發散).

Ex 1: $\int_1^{\infty}\frac{1}{x+1}dx$

Ex 2: $\int_{-\infty}^{\infty} xe^{-x^2}dx$

Ex 3: $\int_{-\infty}^{\infty}\sin x dx$

Ex 4: Find the area of the region R under the curve $y=e^{-\frac{x}{2}}, \forall x\ge 0$.

Case 2: Improper integrals with infinite discontinuities.

(1) If f is continuous on $[a,b)$ and

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx \text{ exists} \Rightarrow \int_a^b f(x)dx \text{ converges.}$$

(2) If f is continuous on $(a,b]$ and

$$\int_a^b f(x)dx = \lim_{s\to a^+}\int_s^b f(x)dx \text{ exists} \Rightarrow \int_a^b f(x)dx \text{ converges.}$$

(3) If f is continuous on $[a,c)\cup(c,b]$ and

$$\int_a^b f(x)dx = \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{s\to c^+}\int_s^b f(x)dx \text{ both limits exist,} \Rightarrow \int_a^b f(x)dx \text{ converges.}$$

Ex 5: $\int_0^1 \frac{1}{x^2}dx$

Ex 6: $\int_{-1}^1 \frac{1}{x}dx$

Ex 7: $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx$

Theorem: Comparison test

Case 1: Let f and g be continuous on $[a,\infty)$ with $0\le f(x)\le g(x), x\ge a$, when

(1) $\int_a^\infty g(x)dx$ converges $\Rightarrow$ $\int_a^\infty f(x)dx$ converges,

(2) $\int_a^\infty f(x)dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)dx$ diverges.

Case 2: Let f and g be continuous on (a,b) with $0\le f(x)\le g(x), x\in(a,b)$, when

(1) $\int_a^b g(x)dx$ converges $\Rightarrow$ $\int_a^b f(x)dx$ converges,

(2) $\int_a^b f(x)dx$ diverges $\Rightarrow$ $\int_a^b g(x)dx$ diverges.

Theorem: Limit comparison test

If the positive functions f and g are continuous on $[a,\infty)$ and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=c$$

(1) $0<c<\infty$ ➔ $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge,

(2) $c=0$ and $\int_a^\infty g(x)dx$ converges ➔ $\int_a^\infty f(x)dx$ converges,

(3) $c=\infty$ and $\int_a^\infty g(x)dx$ diverges ➔ $\int_a^\infty f(x)dx$ diverges.

Test p-function:

(1) $\int_1^\infty \frac{1}{x^p}dx=\begin{cases}\frac{1}{p-1}, & p>1\\ diverges, & p\le 1\end{cases}$

(2) $\int_e^\infty \frac{1}{x(\ln x)^p}dx=\begin{cases}\frac{1}{p-1}, p>1\\ diverges, p\le 1\end{cases}$

(3) $\int_0^1 \frac{1}{x^p}dx=\begin{cases}\frac{1}{1-p}, p<1\\ diverges, p\ge 1\end{cases}$

Ex 8: Determine whether convergent for

(1) $\int_1^\infty \frac{\sin^2\theta}{\theta^2}d\theta$

(2) $\int_0^\infty \frac{1}{e^x+3}dx$

Def: Gamma function

$$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}dx,\ (\alpha>0)$$

Ex 9: Prove that (1) the Gamma function is convergent.

(2) $\Gamma(1)=1$

(3) $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$

(4) $\Gamma(n+1)=n!, (n\in\mathbb{N})$

(5) $\Gamma(\frac{1}{2})=\sqrt{\pi}$, [if $\int_0^{\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$]