8. L'Hôpital Rule and Improper Integrals

8.1 L'Hôpital Rule (羅必達法則)

Theorem: (L'Hôpital Rule)

Let f and g be differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0, \text{ (indeterminate form } \frac{0}{0}\text{)}$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty, \text{ (indeterminate form } \frac{\infty}{\infty}\text{)}$$

➔ $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

If the limit on the right side exists (or is ∞ or $-\infty$).

Ex 1: $\displaystyle \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

Ex 2: $\displaystyle \lim_{x \to 1} \frac{\ln x}{x-1}$

Ex 3: $\displaystyle \lim_{x \to \infty} \frac{x+\sin x}{x}$

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Ex 4: $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^2}$

Concept: $e^{\alpha x} \gg x^{\beta} \gg (\ln x)^{\gamma}$, $as \quad x \to \infty, \forall \alpha, \beta, \gamma > 0$.

Ex 5: $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{\sqrt[3]{x}}$

Indeterminate form $0 \bullet \infty$:

Ex 6: $\lim\limits_{x\to 0^+} x \ln x$

Indeterminate form $\infty - \infty$:

Ex 7: $\lim\limits_{x\to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$

Indeterminate form $0^0, 1^\infty, \infty^0$:

If $f(x) > 0, \forall x \in (a, a+\delta)$ and $\lim\limits_{x\to a^+} f(x)^{g(x)}$ has the forms $0^0, 1^\infty, \infty^0$

$\Rightarrow \lim\limits_{x\to a^+} f(x)^{g(x)} = \lim\limits_{x\to a^+} e^{\ln f(x)^{g(x)}} = e^{\lim_{x\to a^+} \ln f(x)^{g(x)}} = e^{\lim_{x\to a^+} g(x) \ln f(x)}$.

Ex 8: $\lim\limits_{x\to 0^+} x^x$

Ex 9: $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$

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