7.3 Lengths of plane curves

1. Length of a function curve

Theorem: If f is continuously differentiable on $[a,b]$, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$

Ex 1: Find the length of the curve $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

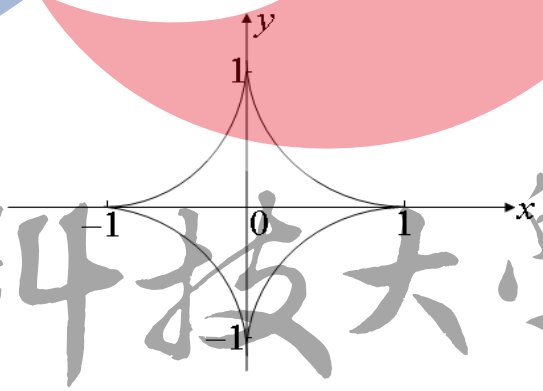
2. Length of a parametric curve

Theorem: If a curve $C:\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t \in [a,b]$. $x'(t)$ and $y'(t)$ are continuous on $[a,b]$, then the length of C is

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt\,.$$

Ex 2: Find the length of the asteroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad t \in [0, 2\pi]$$



3. Length of a polar curve

Theorem: If the curve $r = f(\theta), \theta \in [\alpha, \beta]$ has a continuous first derivative, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r'(\theta))^2}\, d\theta .$$

Ex 3: Find the length of the curve $r = e^{\theta}, \theta \in [0, \pi]$.