7 Applications of definite integrals

7.1 Area between curves

1. If $f(x)$ and $g(x)$ are continuous on $[a,b]$. The area of the region R bounded by the graphs of $y=f(x), y=g(x), x\in[a,b]$ and x-axis is

$$A(R)=\int_a^b |f(x)-g(x)|\,dx$$

Case 1. If $R=\{(x,y)\,|\,a\le x\le b, 0\le x\le f(x)\}$ (See Figure 7.1) , the area is

$$A(R)=\int_a^b f(x)dx$$

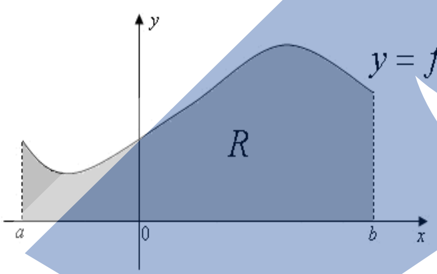

Figure 7.1

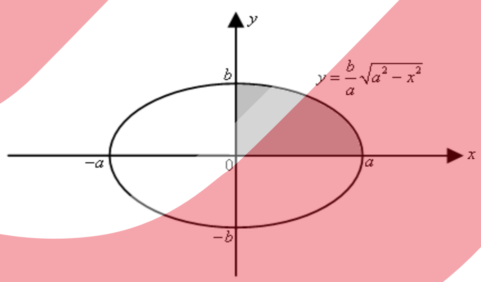

Figure 7.2

Ex1: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (See Figure 7.2).

Case 2. If $R=\{(x,y)\,|\,a\le x\le b, g(x)\le y\le f(x)\}$ (See Figure 7.3), the area is

$$A(R)=\int_a^b f(x)-g(x)dx\,.$$

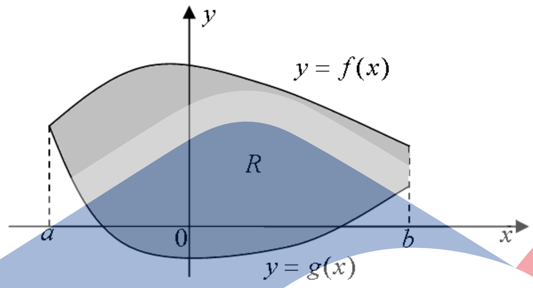

Figure 7.3

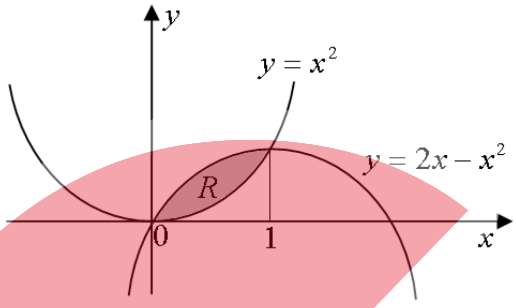

Figure 7.4

Ex2: Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$. (See Figure 7.4)

Case 3. If $R=\{(x,y)\,|\,\phi(y)\le x\le \psi(y), c\le y\le d\}$ (See Figure 7.5), the area is

$$A(R)=\int_c^d \psi(y)-\phi(y)dy\ .$$

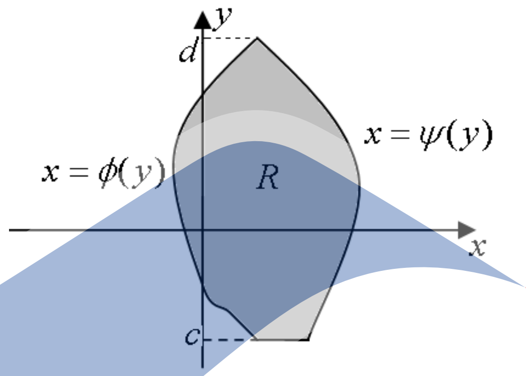

Figure 7.5

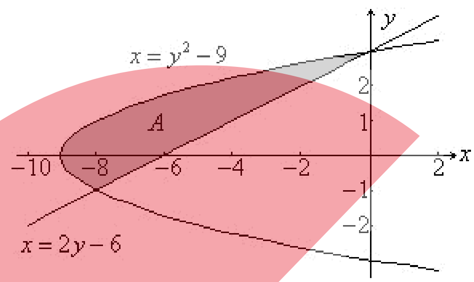

Figure 7.6

Ex 3: Find the area enclosed by the line $x=2y-6$ and the parabola

$x=y^2-9$. (See Figure 7.6)

2. Area in polar coordinate

直角座標 (x,y)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

極座標 (r,θ)

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

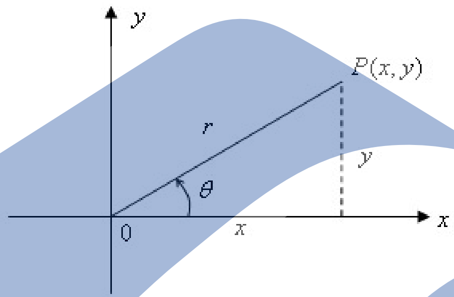

Figure 7.7(a)

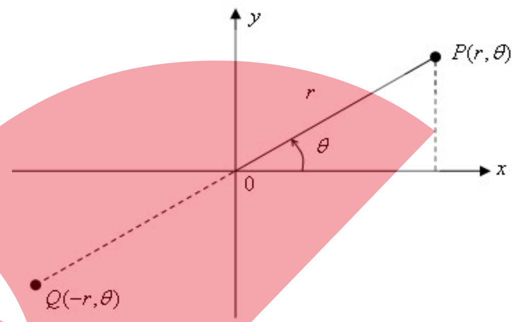

Figure 7.7(b)

Ex 4. Find a Cartesian equivalent for the polar equation and graph the polar curve.

(a) $r = 2\cos\theta$ (b) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Ex 5: Sketch the graph of $r = 1 + \cos\theta$. (See Figure 7.9)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	$1+\frac{\sqrt{3}}{2}$	$1+\frac{\sqrt{2}}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$1-\frac{\sqrt{2}}{2}$	$1-\frac{\sqrt{3}}{2}$	0

Theorem: The area of the region between the origin and the curve $r = f(\theta)$, $\theta \in [\alpha, \beta]$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)d\theta .$$

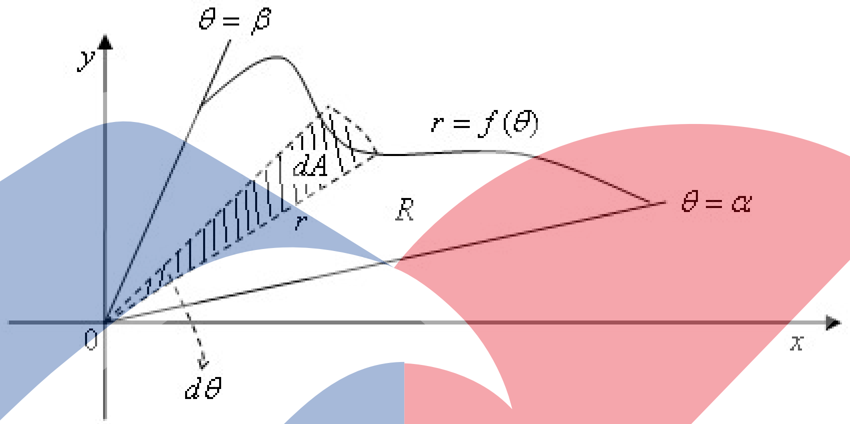

Figure 7.8

Because $dA = \frac{1}{2}r^2 d\theta = \frac{1}{2} f^2(\theta)d\theta$, $A(R) = \int_0^{A(R)} dA = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)d\theta$.

Ex 6: Find the area of the region enclosed by the cardioid $r = 1 + \cos\theta$.

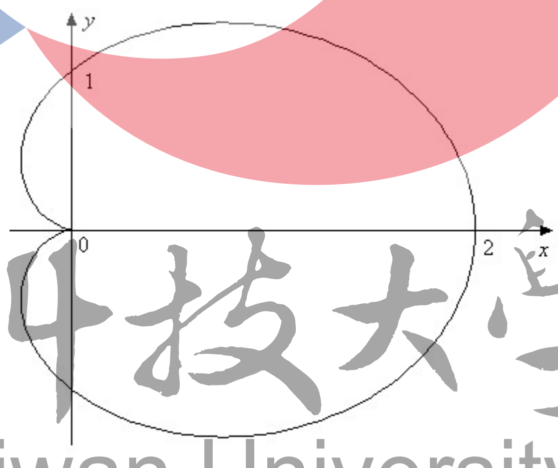

Figure 7.9

3. Area in parametric form

Theorem: If the curve $C:\begin{cases} x=f(t) \\ y=g(t) \end{cases}$, $t\in[\alpha,\beta]$ and $g(t)$, $f'(t)$ are continuous on $[\alpha,\beta]$, then the area under the curve is

$$A=\int_{f(\alpha)}^{f(\beta)} y\,dx=\int_{\alpha}^{\beta} g(t)\,f'(t)\,dt$$

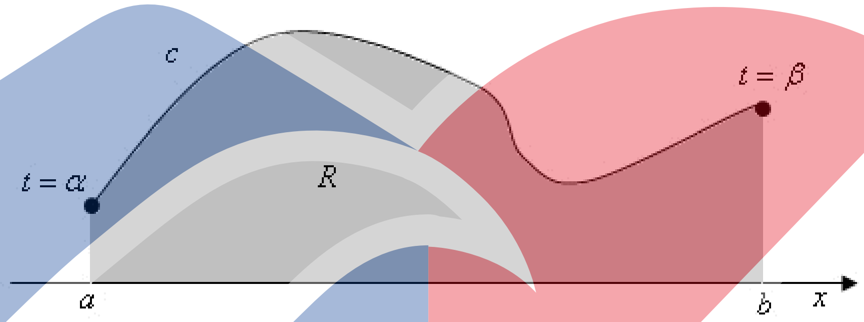

Figure 7.10

Ex 7: Find the area under one arch of the cycloid $\begin{cases} x=r(\theta-\sin\theta) \\ y=r(1-\cos\theta) \end{cases}$, $\theta\in[0,2\pi]$.

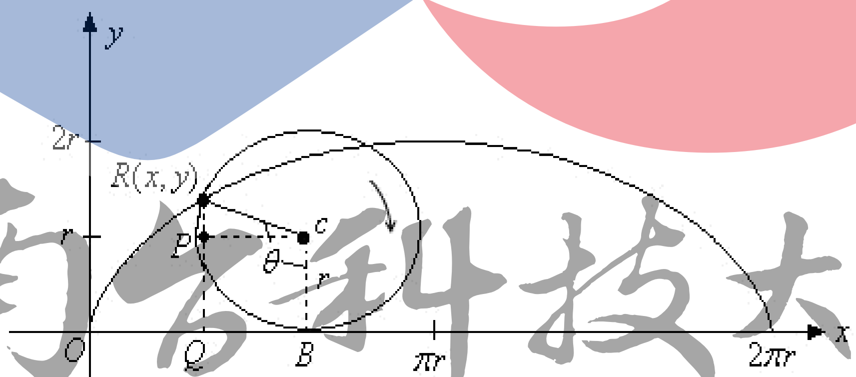

Figure 7.11