6.3 Partial Fractions (部分分式)

If $p(x), g(x) \in$ polynomials

(1) $\deg p(x) \le \deg q(x) \Rightarrow \exists Q(x), r(x) \ni$

$$\frac{q(x)}{p(x)} = Q(x) + \frac{r(x)}{p(x)}$$, [improper fraction=quotient + proper fraction]

where $\deg r(x) < \deg p(x)$, $r(x)$: remainder, $Q(x)$: quotient

Ex1: $\int \frac{x^3+1}{x+2}dx$

(2) $\deg p(x) > \deg q(x)$

<1> if $p(x) = (x-a_1)(x-a_2)\cdots(x-a_n), a_i \ne a_j \forall i \ne j$

$\Rightarrow \exists!\ A_1, A_2, \cdots, A_n \ni$

$$\frac{q(x)}{p(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

How to find $A_i, \forall i$

Method 1: $A_i = \left.\frac{q(x)}{p(x)}(x-a_i)\right|_{x=a_i}$

Method 2: $\because q(x) = \frac{A_1}{x-a_1}p(x) + \frac{A_2}{x-a_2}p(x) + \cdots + \frac{A_n}{x-a_n}p(x)$ is an identical equation.

Let $x = a_i \Rightarrow$ we can obtain A_i, or compare the coefficients of like terms on opposite sides of the equation.

Ex2: $\int \frac{5x+4}{(x+1)(2x-1)}dx$

<2> if $p(x)=(x-a)^n$

$\Rightarrow \exists!\ A_1, A_2, \cdots, A_n \ni$

$$\frac{q(x)}{p(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}$$

Ex3: $\int \frac{x^2-2x+1}{(x+2)^3}dx$

<3> if $p(x)=(ax^2+bx+c)^n$

$\Rightarrow \exists!\ A_1, \cdots, A_n, B_1, \cdots, B_n \ni$

$$\frac{q(x)}{p(x)}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Ex4: $\int \frac{x}{(x+1)^2(x^2+1)}dx$

Ex5: $\int \frac{dx}{x(x^2+1)^2}$