6. Techniques of integration

6.1 Integration by substitution

Ex1: $\int \sqrt{x}dx = \frac{2}{3}x^{3/2} + c$

$$\int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$$

Let $u = 1 + x^2$

$$\int \sqrt{1+x^2}\, d(1+x^2) = \frac{2}{3}(1+x^2)^{3/2} + c$$

$$||$$

$$\int \sqrt{1+x^2}\, 2xdx$$

The substitution method:

1. $\int f(g(x))g'(x)dx$

$= \int f(g(x))dg(x)$

$= \int f(u)du$ $\quad$ $[\![\text{Let } u = g(x)]\!]$

$= F(u) + c$ $\quad$ $[\![\text{If } F'(x) = f(x)]\!]$

$= F(g(x)) + c$

2. $\int_a^b f(g(x))g'(x)dx$

$= \int_a^b f(g(x))dg(x)$

$= \int_{g(a)}^{g(b)} f(u)du$ $\quad$ $[\![\text{Let } u = g(x), \therefore u = g(a) \text{ when } x = a,\ u = g(b) \text{ when } x = b]\!]$

$= F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$

Ex2: $\int \sqrt{2x+1}dx$

Sol. 1: Let $u = 2x + 1$

Sol. 2: Let $u = \sqrt{2x+1}$

Sol. 3: $\int \sqrt{2x+1}dx = \frac{1}{2}\int \sqrt{2x+1}d(2x+1) = \frac{1}{2}\cdot\frac{2}{3}(2x+1)^{3/2} + c = \frac{1}{3}(2x+1)^{3/2} + c$

Ex3: $\int_0^1 x(1-x)^{10}dx$

Sol. 1: Let $u = 1-x$

Sol. 2: $\int_0^1 x(1-x)^{10}dx = \int_0^1 (1-x-1)(1-x)^{10}d(1-x)$

Basic integration formulas:

1. $\int du = u + c$
2. $\int u^r du = \frac{1}{r+1}u^{r+1} + c, r \neq -1$
3. $\int \frac{1}{u}du = \ln|u| + C$
4. $\int e^u du = e^u + C$
5. $\int a^u du = \frac{a^u}{\ln a} + C$
6. $\int \sin u du = -\cos u + C$
7. $\int \cos u du = \sin u + C$
8. $\int \sec^2 u du = \tan u + C$
9. $\int \csc^2 u du = -\cot u + C$
10. $\int \sec u \tan u du = \sec u + C$
11. $\int \csc u \cot u du = -\csc u + C$
12. $\int \tan u du = \ln|\sec u| + c = -\ln|\cos u| + c$
13. $\int \cot u du = \ln|\sin u| + c = -\ln|\csc u| + c$
14. $\int \sec u du = \ln|\sec u + \tan u| + c$

15. $\int \csc u du = \ln|\csc u - \cot u| + c$

16. $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

19. $\int \sinh u du = \cosh u + C$

20. $\int \cosh u du = \sinh u + C$

21. $\int \frac{1}{\sqrt{u^2+a^2}} du = \sinh^{-1}\frac{u}{a} + C,\ u \in \mathbb{R}$

22. $\int \frac{1}{\sqrt{u^2-a^2}} du = \cosh^{-1}\frac{u}{a} + C,\ u > a > 0$

To create differential:

(1) $dx = \frac{1}{a} d(ax+b)$

Ex4: $\int (x-1)^4 dx$

Ex5: $\int e^{-3x} dx$

Ex6: $\int_0^1 \frac{1}{1+5x} dx$

Ex7: Show that $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\frac{u}{a} + C$

Ex8: $\int \frac{1}{\sqrt{8x-x^2}} dx$

Ex9: $\int \frac{1}{\sqrt{x^2+4}} dx$

(2) $x^r dx = \frac{1}{r+1} dx^{r+1}$, $r \neq -1$

Ex10: $\int \frac{x}{\sqrt{1-4x^2}} dx$

Ex11: $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$

Ex12: $\int xe^{-x^2} dx$

Ex13: $\int \frac{2x-9}{\sqrt{x^2-9x+2}} dx$

(3) $\frac{1}{x} dx = d\ln x$, $\frac{1}{u} du = d\ln u$

Ex14: $\int_1^e \frac{\ln x}{x} dx$

(4) $e^x dx = de^x$, $e^u du = de^u$

Ex15: $\int e^{e^x+x} dx$

Ex16: $\int \frac{1}{1+e^x} dx$

(5) $\sin x dx = -d\cos x$, $\cos x dx = d\sin x$

Ex17: $\int \cos x \cdot e^{\sin x} dx$

Ex18: Show that ① $\int \tan x dx = \ln|\sec x| + c$ ② $\int \sec x dx = \ln|\sec x + \tan x| + c$

(6) $\sinh x dx = d \cosh x, \ \cosh x dx = d \sinh x$

Ex19: $\int \cosh 2x \sinh^2 2x dx$

Theorem: Integration of even and odd function

(1) If f is odd function $\Rightarrow \int_{-a}^{a} f(x)dx = 0$ (Fig.6-1)

[f is odd $\Leftrightarrow$ $f(-x) = -f(x)$ $\Leftrightarrow$ the graph of f is symmetric about the origin]

(2) If f is even function $\Rightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ (Fig.6-2)

[f is even $\Leftrightarrow$ $f(-x) = f(x)$ $\Leftrightarrow$ the graph of f is symmetric about the y-axis]

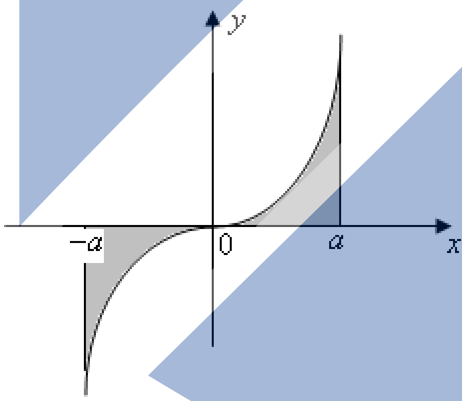

Fig.6-1

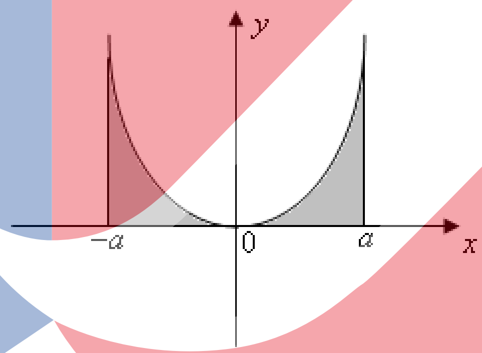

Fig.6-2

Ex20. Evaluate $\int_{-2}^{2} \frac{\sin^3 x}{2 + \cos x} dx$