5.3 Indefinite Integrals

Def: Antiderivatives and indefinite integration

If $\frac{d}{dx}F(x)=f(x)$ (or $dF(x)=f(x)dx$) then $F(x)$ is called the antiderivative of $f(x)$. The set of all antiderivatives of $f(x)$ is called indefinite integration of $f(x)$ and is denoted by

$$\int f(x)dx=F(x)+c,\quad c: \text{constant of integration.}$$

Concept:

(1) Because $(F(x)+c)'=f(x)$, $F(x)+c$ is also indefinite integration of $f(x)$.

(2) $\int d \neq d\int$

$$f(x)\xrightarrow{d} df(x)\xrightarrow{\int} f(x)+c,\quad i.e., \int df(x)=f(x)+c$$

$$f(x)\xrightarrow{\int} F(x)+c\xrightarrow{d} f(x),\quad i.e., \frac{d}{dx}\int f(x)dx=f(x).$$

Ex1: $\int dx=x+c$, $\int d\sin x=\sin x+c$

Ex2: $\frac{d}{dx}\int 3^x dx=3^x$, and $\int d3^x=3^x+c$

(3) Solving a differential equation:

Ex3: If $\frac{dy}{dx}=2x$, and $y(0)=1$, find the particular solution?

Sol: Because $dy=2xdx$, we have

$$\int dy=\int 2xdx.\quad [\![\because (x^2)'=2x]\!]$$

So, $y=x^2+c$ (General solution)

Substitute the initial condition $y(0)=1$ to get

$1=0^2+c, \therefore c=1$.

Thus, we obtain the particular solution $y=x^2+1$.

(4) If $F'(x)=f(x)$, <1> $\int f(x)dx=F(x)+c$ <2> $\int_a^b f(x)dx=F(x)\Big|_{x=a}^{x=b}=F(b)-F(a)$

Ex4: $\because (3x)'=3, \therefore \int 3dx=3x+c$, and $\int_1^4 3dx=3x\Big|_1^4=3(4-1)=9$

Formula:

(1) $\int kf(x)dx = k\int f(x)dx, \forall k \in \mathbb{R}$

Ex5: $\int kdx = kx + c$

(2) $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Ex6: $\int 3+2xdx = \int 3dx + \int 2xdx = 3x + x^2 + c$

Differentiation Formula

(1) $d(\frac{1}{r+1}x^{r+1}) = x^r dx, (r \neq -1)$

Integration Formula

$\int x^r dx = \frac{1}{r+1}x^{r+1} + c, \quad (r \neq -1)$

Ex7: <1> $\int 1dx = \int x^0 dx = \frac{1}{1}x^{0+1} + c = x + c$,

<2> $\int xdx = \frac{1}{2}x^2 + c$,

<3> $\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_{x=0}^{x=1} = \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$

Ex8: $\int 3x + 2dx$

Ex9: $\int t^3 dt$

Ex10: $\int \frac{1}{x^3}dx$

Ex11: $\int \frac{x^2+2x}{\sqrt{x}}dx$

Ex12: $\int_0^1 |2x-1|\,dx$

Ex13: $\int_0^4 [x]dx$

(2) $de^x = e^x dx$

$\int e^x dx = e^x + c$

$da^x = a^x \ln a dx$ $\quad\quad$ $\int a^x dx = \frac{a^x}{\ln a} + c$

Ex14: $\int_0^1 e^x dx$ $\quad\quad$ Ex15: $\int 3^x dx$

(3) $d \ln|x| = \frac{1}{x} dx$ $\quad\quad$ $\int \frac{1}{x} dx = \ln|x| + c$

Ex16: $\int_1^e \frac{x^2+1}{x} dx$

(4) $d \cos x = -\sin x dx$ $\quad\quad$ $\int \sin x dx = -\cos x + c$

$d \sin x = \cos x dx$ $\quad\quad$ $\int \cos x dx = \sin x + c$

$d \tan x = \sec^2 x dx$ $\quad\quad$ $\int \sec^2 x dx = \tan x + c$

$d \cot x = -\csc^2 x dx$ $\quad\quad$ $\int \csc^2 x dx = -\cot x + c$

$d \sec x = \sec x \tan x dx$ $\quad\quad$ $\int \sec x \tan x dx = \sec x + c$

$d \csc x = -\csc x \cot x dx$ $\quad\quad$ $\int \csc x \cot x dx = -\csc x + c$

Ex17: $\int \sin x + 2\cos x dx$ $\quad\quad$ Ex18: $\int_0^{\pi/2} \sin x dx$

Ex19: $\int (\sec x + \tan x)^2 dx$

(5) $d \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} dx$ $\quad\quad$ $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$d \tan^{-1} x = \frac{1}{1+x^2} dx$ $\quad\quad$ $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$d \sec^{-1}|x| = \frac{1}{x\sqrt{x^2-1}} dx$ $\quad\quad$ $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}|x| + c$

Ex20: $\int_0^{1/2} \frac{1}{\sqrt{1-u^2}} du$

(6) $d\cosh x = \sinh x dx$ $\qquad$ $\int \sinh x dx = \cosh x + c$

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Ex21: $\int_0^1 \dfrac{1}{\sqrt{1+x^2}} dx$

Ex22: If $\dfrac{dy}{dt} = ky$ for some $k \in \mathbb{R}$, with the initial condition $y(0) = y_0$. Find y