5.2 The Fundamental Theorem of Calculus

Let $f(x)$ be continuous on $[a,b]$ and let $A(x)$ be the area function of the region bounded by $y=f(x)$, the x-axis, and variable x from a to x. i.e.,

$$A(x)=\int_a^x f(t)dt.$$

Question: What is $A'(x)$?

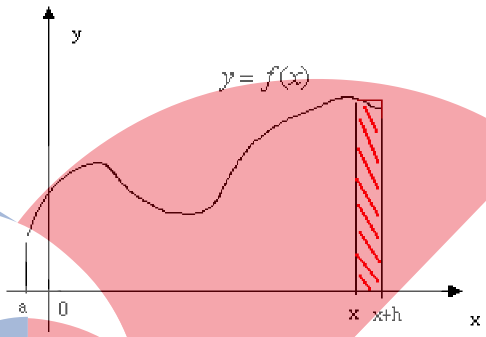
Consider $h\approx 0$, then

$$A(x+h)-A(x)\approx h\bullet f(x)$$

$$\Rightarrow \frac{A(x+h)-A(x)}{h}\approx f(x)$$

$$\Rightarrow \lim_{h\to 0}\frac{A(x+h)-A(x)}{h}=f(x)$$

$$\Rightarrow A'(x)=f(x),\quad \forall x\in(a,b).$$



The Second Fundamental Theorem of Calculus:

If f is continuous on $[a,b]$, then

$$\frac{d}{dx}\int_a^x f(t)dt=f(x),\quad \forall x\in(a,b).$$

Ex1: Find $\frac{d}{dx}\int_1^x \sqrt{1+t^2}\,dt$

Theorem: If $F'(x)=A'(x)=f(x),\ \forall x\in(a,b)\Rightarrow \exists\ c\in\mathbb{R},\ni$

$$A(x)=F(x)+c,\quad \forall x\in(a,b).$$

Lemma: (1) $A(a)=\int_a^a f(x)dx=0$

(2) $A(b)=\int_a^b f(x)dx=A(b)-A(a)=F(b)-F(a)$

So, $\int_a^b f(x)dx=F(b)-F(a)=F(x)\Big|_{x=a}^{x=b}$

The First Fundamental Theorem of Calculus:

If f is continuous on $[a,b]$ and $F'(x)=f(x),\ \forall x\in(a,b)\Rightarrow$

$$\int_a^b f(x)dx=F(x)\Big|_{x=a}^{x=b}=F(b)-F(a).$$

Ex2: $\int_0^1 x\,dx$

Ex3: $\int_0^1 e^x\,dx$

Ex4: If $f'(x) = g(x),\ f(-1) = 2,\ f(3) = 7$, find $\int_{-1}^{3} g(x)\,dx$ and $\int_{3}^{-1} 2g(u)\,du$

Theorem:

(1) $\dfrac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))h'(x)$

(2) $\dfrac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

Ex5: Show that the function $f(x) = \int_1^{\sqrt{x}} e^{y^2}\,dy,\ \forall x > 0$ is increasing.

Ex6: Find $\dfrac{d}{dx}\int_{x^2}^{x^3} \sin(2+t^3)\,dt$