5. Integration

5.1 Area and the definite integral

Def: Let $f(x)$ be defined on $[a,b]$, and let Δ be a partition of $[a,b]$ given by $\Delta: a = x_0 < x_1 < \cdots < x_n = b$. For any $\xi_i \in [x_{i-1}, x_i]$, the sum

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

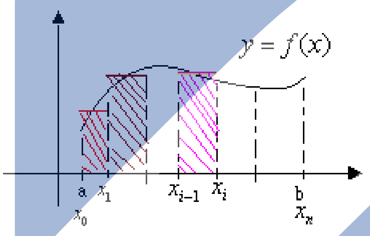
is called a Riemann sum of $f(x)$ for the partition Δ on $[a,b]$. If the limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = s\,, \quad \|\Delta\| = \max_{1\le i\le n} |\Delta x_i|$$

exists. We say that f is integrable on $[a,b]$ and the limit

$$s = \int_a^b f(x)dx = \int_{\text{lowerlimit}}^{\text{upperlimit}} (\text{integrand})(\text{differential})$$

is called the definite integral of f from a to b.



Concept:

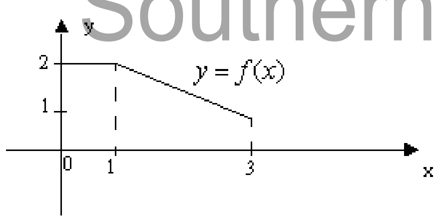
(1) If $f(x) \ge 0, \forall x \in [a,b]$, then

$\int_a^b f(x)dx =$ the area of the region bounded by the graph of f, $x = a$, $x = b$, and x-axis.

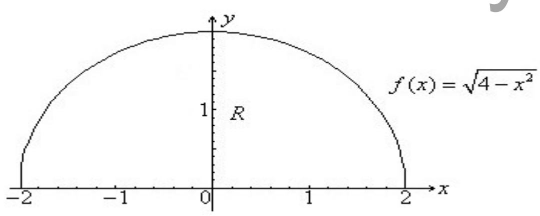
Ex1: $\int_0^1 xdx$

Ex2: $\int_0^1 3dx$

Ex3: Find $\int_0^3 f(x)dx$, if



Ex4: $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$



(2) "$\int$" was first used by Leibniz (1646-1716). The meaning is "Summa".

(3) If $f(x)=\begin{cases}1, x\in\mathbb{Q}\\ 0, x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$, then $\int_0^1 f(x)dx$ not exists.

(4) If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Operation of definite integrals:

If $f(x)$ and $g(x)$ are integrable on $[a,b]$, then

(1) $\int_a^b kf(x)dx = k\int_a^b f(x)dx,\ \ \forall k\in\mathbb{R}$.

Ex5: $\int_0^1 3xdx = 3\int_0^1 xdx$

(2) $\int_a^b f(x)\pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

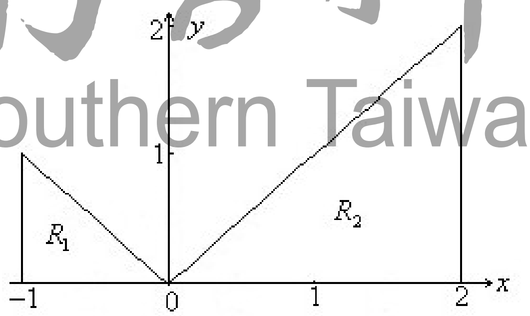
Ex6: $\int_0^1 3+xdx = \int_0^1 3dx + \int_0^1 xdx$

(3) $\int_a^a f(x)dx = 0$.

Ex7: $\int_1^1 3+xdx = 0$

(4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ \ \forall c\in[a,b]$.

Ex8: $\int_{-1}^2 |x|dx = \int_{-1}^0 |x|dx + \int_0^2 |x|dx = R_1 + R_2$



(5) $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Ex9: If $\int_2^1 f(x)dx = 4, \int_1^7 f(x)dx = 6$. Find $\int_1^2 f(x)dx$ and $\int_2^7 f(x)dx$

Preservation of definite integrals:

(1) If $f(x) \ge 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)dx \ge 0$.

(2) If $f(x) \ge g(x)$, $\forall x \in [a,b]$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

(3) If $f(x)$ is continuous on $[a,b]$, then $\exists$ $M = \max_{a \le x \le b} f(x)$, $m = \min_{a \le x \le b} f(x)$, such that

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Ex10: Show that $1 \le \int_0^1 e^x dx \le e$.

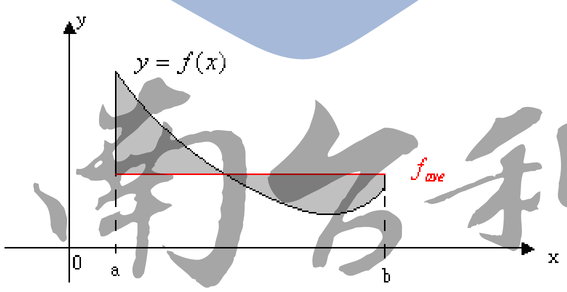
Mean Value Theorem for Integrals:

If $f(x)$ is continuous on $[a,b]$, then $\exists c \in [a,b]$ $\ni$

$$\int_a^b f(x)dx = f(c)(b-a)$$

or $\frac{1}{b-a}\int_a^b f(x)dx = f(c) \equiv f_{ave}$ (the average value of $f(x)$ on $[a,b]$)

$\left(\frac{area}{width} = average \quad heigth\right)$



Ex11: Find the average value of $f(x) = \sqrt{4-x^2}$ over the interval $[-2,2]$ and find

$c \in [-2,2]$ $\ni$ $f(c) = f_{ave}$