11.3 Triple integrals

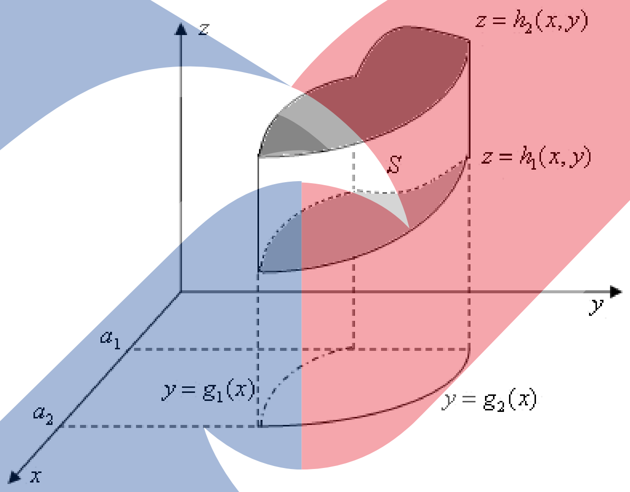
Theorem:

Let f be continuous on

$$S = \{(x,y,z) \mid a_1 \le x \le a_2, g_1(x) \le y \le g_2(x), h_1(x,y) \le z \le h_2(x,y)\}.$$

Then

$$\iiint_S f(x,y,z)dV = \int_{a_1}^{a_2}\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx$$



Concept: (1) If $f(x,y,z) = \rho(x,y,z)$: density, $\forall (x,y,z) \in S$. Then

$$\iiint_S f(x,y,z)dV = \text{mass of } S$$

(2) $\iiint_S dV = |S|$, (volume of S)

Ex 1: Find the volume of the region S enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Ex 2: Let S be a solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by the plane $z=4-2x$. Evaluate $\iiint_S \sqrt{2x-x^2}\,dxdydz$.

Change of variables for triple integrals

Theorem: If $f(x,y,z)$ is continuous on S. $T:S^* \to S$ is 1-1 and

$$T:\begin{cases} x=g(u,v,w) \\ y=h(u,v,w) \\ z=k(u,v,w) \end{cases}$$

where g,h and k have continuous partial derivatives on S^* and the Jacobian

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0, (u,v,w)\in S^*.$$

Then

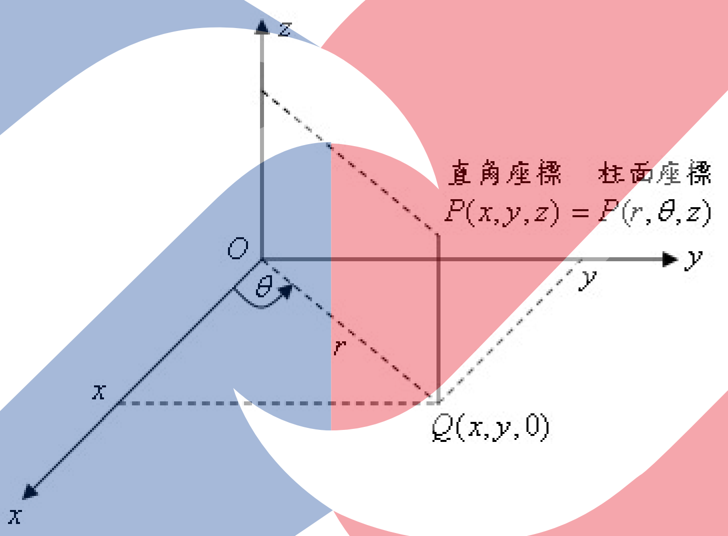
$$\iiint_S f(x,y,z)\,dxdydz$$

$$=\iiint_{S^*} f(g(u,v,w),h(u,v,w),k(u,v,w))\left|J(u,v,w)\right|dudvdw.$$

1. Triple integrals in cylindrical coordinates

Rectangular coordinate (x, y, z) — Cylindrical coordinate (r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\frac{y}{x} \\ z = z \end{cases}$$



Theorem: If $f(x, y, z)$ is continuous on S. $T : S^* \to S$ and

$$T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta, (r, \theta, z) \in S^*. \\ z = z \end{cases}$$

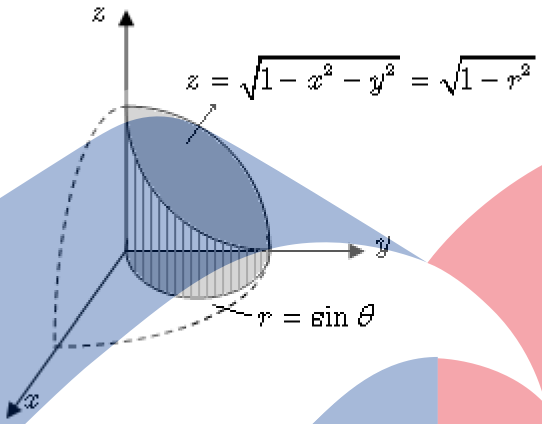
Then the Jacobian

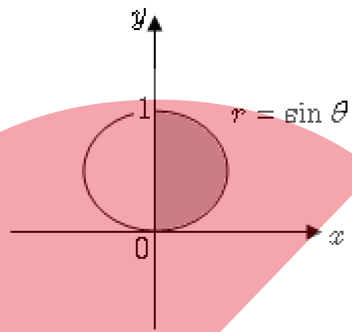
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

and

$$\iiint_S f(x, y, z) = \iiint_{S^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

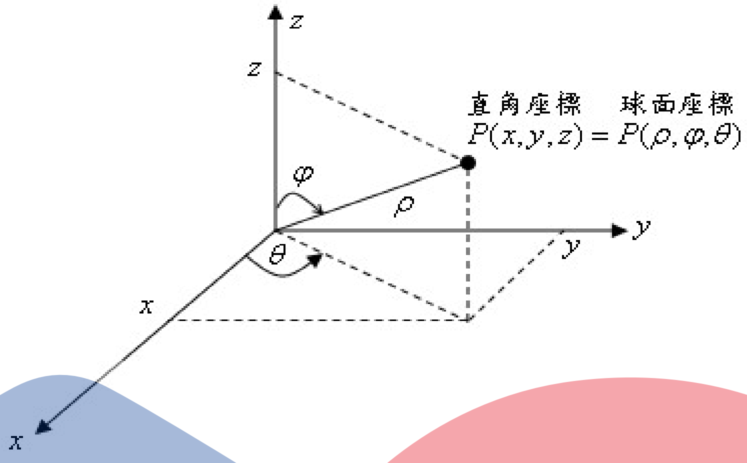
Ex 3: Find the volume of the solid region S cut from the sphere $x^2+y^2+z^2=1$ by the cylinder $x^2+y^2=y$.





2. Triple integrals in spherical coordinates

Rectangular coordinate	Spherical coordinate
(x,y,z)	(ρ,φ,θ)
$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$	$\begin{cases} \rho=\sqrt{x^2+y^2+z^2} \\ \varphi=\cos^{-1}\frac{z}{\rho} \\ \theta=\tan^{-1}\frac{y}{x} \end{cases}$



Theorem: If $f(x,y,z)$ is continuous on S. $T: S^* \to S$ and

$$T: \begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta\,, \quad (\rho,\varphi,\theta)\in S^*. \\ z = \rho\cos\varphi \end{cases}$$

Then the Jacobian

$$J(\rho,\varphi,\theta)=\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}=\begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta} \end{vmatrix}$$

$$=\begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}=\rho^2\sin\varphi$$

and

$$\iiint_S f(x,y,z)\,dxdydz=\iiint_{S^*} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Ex 4: Find the volume of the solid region S bounded by the cone $z^2=x^2+y^2$ and above by the sphere $x^2+y^2+z^2=9$.