11.2 Change of variables for double integrals

Theorem:

If f is continuous on Ω. Let the change of variable $T:\Omega^* \to \Omega$ be 1-1 function, and

$$T:\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

where g, h have continuous partial derivatives on Ω^*, and the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} \neq 0 .$$

Then

$$\iint_{\Omega} f(x,y)dxdy = \iint_{\Omega^*} f(g(u,v),h(u,v))\,|J|\,dudv$$

Concept: (1) J 為面積的增倍量，所以取正。

(2) $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

1. Polar coordinate transformation

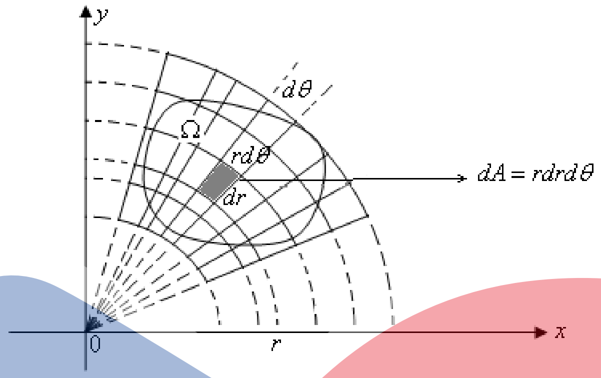
$$T:\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, (r \geq 0, \theta_0 \leq \theta < \theta_0 + 2\pi)$$, then

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r ,$$

and

$$\iint_{\Omega} f(x,y)dxdy = \iint_{\Omega^*} f(r\cos\theta, r\sin\theta)rdrd\theta .$$

Concept: $dA = dxdy = rdrd\theta$



Ex 1: Find the volume of the solid region bounded above by the hemisphere

$z = \sqrt{16 - x^2 - y^2}$ and below by the circular region $x^2 + y^2 \le 4$.

Ex 2: Find $\int_0^{\infty} e^{-x^2} dx$

2. Linear transformation

If $T:\begin{cases} x = au + bv \\ y = cu + dv \end{cases}$ and $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, then

$$\iint_{\Omega} f(x,y)\,dxdy = \iint_{\Omega^*} f(au+bv, cu+dv)\,|ad-bc|\,dudv .$$

Ex 3: Evaluate $\iint_{\Omega} (x+y)^2 dxdy$, where Ω is the region bounded by the lines

$x+y=0, x+y=1, 2x-y=0$ and $2x-y=3$.

Ex 4: Evaluate $\iint_{\Omega} x^2 y^2 dxdy$, where Ω is the region enclosed by

$xy=1, xy=2, y=4x$ and $y=x$.