10.7 Extreme Values for functions of several variables

Define: (1) If $f(x,y) \le f(a,b),\ \forall (x,y) \in D_f$ ➔ $f(a,b)$ is an absolute maximum.

If $f(x,y) \ge f(a,b),\ \forall (x,y) \in D_f$ ➔ $f(a,b)$ is an absolute minimum.

(2) If $f(x,y) \le f(a,b)$ when (x,y) is near (a,b) ➔ $f(a,b)$ is a relative (local) maximum.

If $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b) ➔ $f(a,b)$ is a relative (local) minimum.

(3) If $f_x(a,b)=0$ and $f_y(a,b)=0$ (i.e. $\nabla f(a,b)=0$) or if one of $f_x(a,b)$ and $f_y(a,b)$ does not exist. Then (a,b) is called a critical point of f.

(4) If (a,b) is a critical point of f, but $f(a,b)$ is not a local maximum or minimum. The point (a,b) is called a saddle point(鞍點) of f.

Ex 1: $f(x,y)=xy$

Theorem: If f has a local maximum or minimum at (a,b) and $f_x(a,b)$, $f_y(a,b)$ exist, then $f_x(a,b)=f_y(a,b)=0$.

Ex 2: If $f(x,y)=x^2+y^2+1$, find critical point and extreme value(s).

1. Second Derivatives Test

Theorem: Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disc with center (a,b), and $f_x(a,b)=f_y(a,b)=0$ (i.e. (a,b) is a critical point). Let

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\cdot f_{yy} - f_{xy}{}^2$$

and $D=D(a,b)$.

(1) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

(2) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

(3) If $D<0$, then (a,b) is a saddle point of f.

(4) If $D=0$, the test is inconclusive.

Ex 3: Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

2. Lagrange Multipliers

(1) Claim: To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

step 1. Let $F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$

step 2. Find all values of x, y, z and λ such that

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_z = 0 \\ F_\lambda = 0 \end{cases}$$

step 3. The largest of these values is the maximum value of f; the smallest is the minimum.

Ex 4: Find the greatest and smallest values of the function

$$f(x, y) = xy$$

Take on the ellipse

$$x^2 + 4y^2 = 8.$$

Ex 5: A rectangular box without a lid is to be made from $12\ \text{m}^2$ of cardboard. Find the maximum volume of such a box.

(2) Claim: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$.

step 1. Let $F(x,y,z,\lambda_1,\lambda_2)=f(x,y,z)+\lambda_1 g_1(x,y,z)+\lambda_2 g_2(x,y,z)$

step 2. Find all values of x,y,z and λ_1,λ_2 such that

$$\begin{cases} F_x=0 \\ F_y=0 \\ F_z=0 \\ F_{\lambda_1}=0 \\ F_{\lambda_2}=0 \end{cases}$$

step 3. The largest of these values is the maximum value of f; the smallest is the minimum.

Ex 6: Find the maximum value of the function $f(x,y,z)=x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.