10.6 Directional derivatives and the Gradient vector

Define: Gradient (梯度)

(1) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ is the gradient of $f(x,y)$.

(2) $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$ is the gradient of $f(x,y,z)$.

Define: Directional derivative (方向導數)

The directional derivative of f at $P(x_0,y_0)$ in the direction of a unit vector $\vec{u} = \langle a,b \rangle$ is

$$D_{\vec{u}} f(P) = \lim_{h\to 0} \frac{f(P+h\vec{u}) - f(P)}{h} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

if this limit exists.

Theorem: If f is differentiable in an open region containing $P(x_0,y_0)$, then

$$D_{\vec{u}} f(P) = \nabla f(P) \bullet \vec{u} = \|\nabla f(P)\| \cos\theta,$$

where θ is the angle between $\nabla f(P)$ and $\vec{u}$.

Concept: (1) The maximum value (increases fastest) of the directional derivative $D_{\vec{u}} f(P)$ is $\|\nabla f(P)\|$ and it occurs when $\vec{u}$ has the same direction as $\nabla f(P)$.

(2) The minimum value (decreases fastest) of the directional derivative $D_{\vec{u}} f(P)$ is $-\|\nabla f(P)\|$ and it occurs when $\vec{u}$ and $\nabla f(P)$ have opposite direction.

Ex 1: (a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2},2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Theorem: Suppose that the surface $S=\{(x,y,z)\mid F(x,y,z)=c\}$. If $P(x_0,y_0,z_0)\in S$, then $\nabla F(P)$ is perpendicular to the level surface at P.

Define:

(1) The tangent plane (切平面) to the level surface S at P as the plane that passes through P and has normal vector (法向量) $\nabla F(P)$.

(2) The normal line (法線) to S at P is the line passing through P and perpendicular to the tangent plane.

Theorem: (1) The tangent plane to the surface $F(x,y,z)=c$ at $P(x_0,y_0,z_0)$ is

$$\nabla F(P)\bullet\langle x-x_0,y-y_0,z-z_0\rangle=0$$

$$\Leftrightarrow F_x(P)(x-x_0)+F_y(P)(y-y_0)+F_z(P)(z-z_0)=0$$

(2) The normal line to $F(x,y,z)=c$ at P is

$$\frac{x-x_0}{F_x(P)}=\frac{y-y_0}{F_y(P)}=\frac{z-z_0}{F_z(P)}$$

or

$$x=x_0+F_x(P)t,\ y=y_0+F_y(P)t,\ z=z_0+F_z(P)t,\ t\in\mathbb{R}$$

Ex 2: Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$.