10.5 Chain Rules for Functions of Several Variables

Assume all following functions are differentiable.

Theorem 1. If $z = f(x,y),\quad x = g(t), y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Ex 1: $z = x^2y^3 - x^4, x = e^{3t}, y = \ln t$ find $\dfrac{dz}{dt}$

Theorem 2. If $z = f(x,y),\quad x = g(s,t), y = h(s,t)$ then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex 2: $z = 3x^2y + xe^y, x = r\cos\theta, y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}, \dfrac{\partial z}{\partial \theta}$.

Theorem 3. If $z = f(w), w = g(x,y)$, then

$$\frac{\partial z}{\partial x} = \frac{df}{dw}\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{df}{dw}\frac{\partial w}{\partial y}$$

Ex 3: If $z = f(x-y)$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

Theorem 4. If $F(x,y)=c, c:\text{constant}$, and $y=y(x)$, then

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

Ex 4: If $x^3+y^3=3xy$, find $\frac{dy}{dx}$.

Theorem 5. If $F(x,y,z)=c, c:\text{constant}$ and $z=f(x,y)$, then

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

Ex 5: If $e^{xyz}=2$, find $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

Theorem 6: If $F(x)=\int_{g(x)}^{h(x)} f(x,y)dy$, then

$$F'(x)=\int_{g(x)}^{h(x)} f_x(x,y)dy+f(x,h(x))\cdot h'(x)-f(x,g(x))\cdot g'(x)$$

Ex 6: If $F(x)=\int_{x}^{x^3}\sqrt{x^2+t^2}\,dt$, find $F'(x)$