10.4 Differentials

Define: (1) If $z = f(x, y)$ is differentiable at (x_0, y_0)

$\Leftrightarrow \exists \varepsilon_1, \varepsilon_2 \to 0$, as $(\triangle x, \triangle y) \to (0,0)$ $\ni$

$$\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x, y) = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y + \varepsilon_1 \triangle x + \varepsilon_2 \triangle y .$$

(2) If $\triangle x = dx, \triangle y = dy$, then the total differential of $z = f(x, y)$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy .$$

(3) Approximation by differentials:

$$f(x + \triangle x, y + \triangle y) \approx f(x, y) + f_x(x, y)\triangle x + f_y(x, y)\triangle y .$$

(4) The approximate change in z :

$$\triangle z = f(x + \triangle x, y + \triangle y) - f(x, y) \approx \frac{\partial f}{\partial x}(x, y)\triangle x + \frac{\partial f}{\partial y}(x, y)\triangle y .$$

Note: If $f(x, y)$ is differentiable at (x_0, y_0) $\Leftrightarrow$ $\displaystyle\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - df}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$

Ex 1: Show that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$.

Ex 2: Let $z = 2x^2 y + y^3$, find

(1) dz

(2) The approximate change in z when x changes from $x = 1$ to $x = 1.01$ and y changes from $y = 2$ to $y = 1.98$.

Ex 3: Using the total differential to approximate $\sqrt{(4.01)^2+(2.98)^2}$.

Theorem: Sufficient condition for differentiability.

If $z=f(x,y)$, f_x and f_y are continuous in an open region R, then f is differentiable on R.

Theorem: Differentiability implies continuity.

If $f(x,y)$ is differentiable at (x_0,y_0), then it is continuous at (x_0,y_0).

Ex 4: If $f(x,y)=\begin{cases}\dfrac{-3xy}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$. Show that $f_x(0,0)$ and $f_y(0,0)$ both exist, but that f is not differentiable at $(0,0)$.