10.2 Limits and continuity

Define: Let $f(x,y)$ be defined except possibly at (x_0,y_0) on an open disc centered at (x_0,y_0), then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L \Leftrightarrow \quad \forall \varepsilon>0, \exists \delta>0, \ni \text{ if } 0<|(x,y)-(x_0,y_0)|<\delta$$
$$\Rightarrow |f(x,y)-L|<\varepsilon$$

Ex 1: $\lim_{(x,y)\to(1,2)} \dfrac{3xy^2}{x^2+y^2}$

Ex 2: $\lim_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^2}$

Concept: If $x=r\cos\theta, y=r\sin\theta$, then $(x,y)\to(0,0) \Leftrightarrow r\to 0$.

Ex 3: $\lim_{(x,y)\to(0,0)} \dfrac{1}{x^2+y^2}$

Ex 4: $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

Define: If $f(x,y)$ is continuous at (x_0,y_0)

$$\Leftrightarrow \lim_{(x,y)\to(x_0.y_0)} f(x,y)=f(x_0,y_0)$$

$$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0, \ni \text{ if } |(x,y)-(x_0,y_0)|<\delta$$
$$\Rightarrow |f(x,y)-f(x_0,y_0)|<\varepsilon$$

Ex 5: Discuss the continuity of $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.