Chapter 4 Applications of derivatives

4.1 Extreme values of functions

Def: Absolute extrema(絕對極值)

Let $f(x)$ be defined on an interval I containing c.

(1) If $f(c) \le f(x),\ \forall x \in I$ ➔ $f(c)$ is the absolute minimum (minimum；絕對極小值) of f on I.

(2) If $f(c) \ge f(x),\ \forall x \in I$ ➔ $f(c)$ is the absolute maximum (maximum；絕對極大值) of f on I.

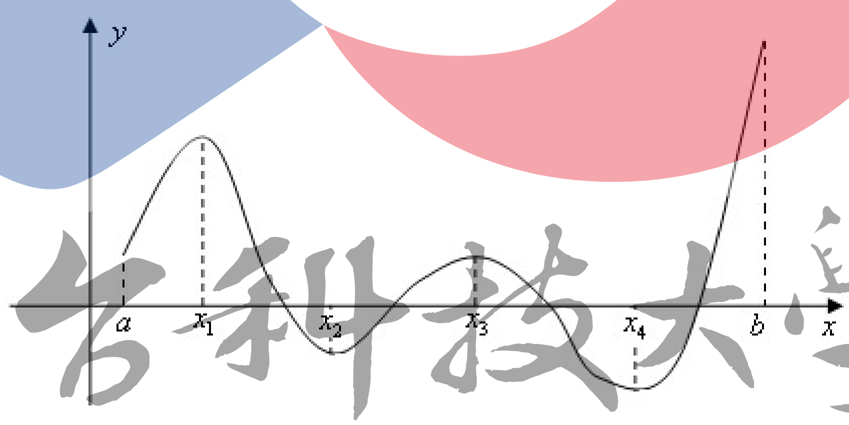
Def: Relative extrema(相對極值)

If $\exists\ \delta > 0$, $\ni$

(1) $f(c) \le f(x),\ \forall x \in (c-\delta, c+\delta)$ ➔ $f(c)$ is called a relative (or local) minimum (相對或局部極小值) of f.

(2) $f(c) \ge f(x),\ \forall x \in (c-\delta, c+\delta)$ ➔ $f(c)$ is called a relative (or local) maximum (相對或局部極大值) of f.

Ex 1: $y = f(x), x \in [a,b]$



The absolute minimum: $f(x_4)$ ； the absolute maximum: $f(b)$

Relative minimum: $f(x_2)$ and $f(x_4)$ ；relative maximum: $f(x_1)$ and $f(x_3)$ 。

Ex 2: $f(x) = \dfrac{1}{x}, x \in (0,2]$

Theorem: The extreme value theorem(極值定理)

If f is continuous on $[a,b]$ ➔ f has both a minimum and a maximum on $[a,b]$.

i.e., $\exists x_1, x_2 \in [a,b] \ni f(x_1) \le f(x) \le f(x_2), \ \forall x \in [a,b]$.

Fermat's Theorem:

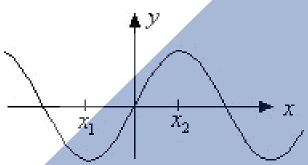
If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

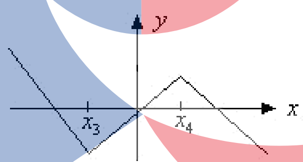
Concept: The converse proposition is wrong.

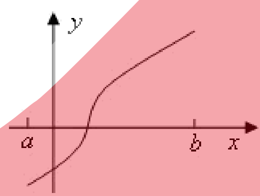
Ex 3: $f(x)=x^3$

Concept: The extrema probably occur at:

(1) $f'(c)=0$ (2) $f'(c)$ does not exist. (3) The endpoints.







Def: Critical point(臨界點)

If $f'(c)=0$ or $f'(c)$ does not exist. Then c is a critical point of f.

Theorem: (Local extrema occur only at critical point)

If f has a local maximum or minimum at c, then c is a critical point.

Concept: To find the absolute maximum and minimum values of a continuous function f on $[a,b]$:

Step 1. Find the critical points of f in (a,b).

Step 2. Evaluate f at each critical point in (a,b).

Step 3. Evaluate f at each endpoint of $[a,b]$.

Step 4. The largest of these values is the maximum; the smallest of these values is the minimum.

Ex 4: Find the extrema of $f(x)=3x^4-4x^3$ on $[-1,2]$.

Ex 5: Find the absolute maximum and minimum values of $f(x)=x^{2/3}$ on $[-2,8]$.

Rolle's theorem:

(1) $f(x)$ is continuous on $[a,b]$.

(2) $f(x)$ is differentiable on (a,b).

(3) $f(a)=f(b)$.

➔ $\exists\ c\in(a,b),\ \ni f'(c)=0$.

Ex 6: Prove that if f is differentiable and $f'(x)<1\ \ \forall x\in\mathbb{R}$, then f has at most one fixed point.

Concept: If $f(c)=c$, ➔ c is called a fixed point of f.

The Mean Value Theorem(均值定理):

(1) $f(x)$ is continuous on $[a,b]$.

(2) $f(x)$ is differentiable on (a,b).

➔ $\exists\ c \in (a,b),\ \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Note: Slope of tangent line = Slope of secant line.

Ex 7: If $f(x) = 5 - \dfrac{4}{x}$, find each value of c in $[1,4]$ such that the f satisfies the Mean Value Theorem.