3.6 Derivatives of trigonometric functions and hyperbolic functions

1. Derivatives of trigonometric functions

Theorem: (1) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(2) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

Ex 1: $\lim_{x\to 0}\frac{\sin 2x}{x}$

Ex 2: $\lim_{x\to 0}\frac{1-\cos(2\sin x)}{x^2}$

Theorem:

(1) $\frac{d}{dx}\sin x=\cos x$ $\quad [\,(\sin u)'=(\cos u)\cdot u'\,]$

(2) $\frac{d}{dx}\cos x=-\sin x$ $\quad [\,(\cos u)'=-\sin u\cdot u'\,]$

(3) $\frac{d}{dx}\tan x=\sec^2 x$ $\quad [\,(\tan u)'=\sec^2 u\cdot u'\,]$

(4) $\frac{d}{dx}\cot x=-\csc^2 x$ $\quad [\,(\cot u)'=-\csc^2 u\cdot u'\,]$

(5) $\frac{d}{dx}\sec x=\sec x\cdot\tan x$ $\quad [\,(\sec u)'=\sec u\cdot\tan u\cdot u'\,]$

(6) $\frac{d}{dx}\csc x=-\csc x\cdot\cot x$ $\quad [\,(\csc u)'=-\csc u\cdot\cot u\cdot u'\,]$

Ex 3: If $f(x) = x\sin^3 x$, find $f'(1)$

Ex 4: Find y' if (a) $y = \sin 2x$ (b) $y = \cos(x^2+1)$ (c) $y = \tan\dfrac{x-1}{x+1}$

(d) $y = \dfrac{\sec x}{1+\tan x}$

2. Derivatives of inverse trigonometric function

Theorem:

(1) $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, $\forall\, |x|<1$ $\quad [(\sin^{-1}u)' = \dfrac{1}{\sqrt{1-u^2}}u']$

(2) $\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$, $|x|<1$ $\quad [(\cos^{-1}u)' = \dfrac{-1}{\sqrt{1-u^2}}u']$

(3) $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$, $x\in\mathbb{R}$ $\quad [(\tan^{-1}u)' = \dfrac{1}{1+u^2}u']$

(4) $\dfrac{d}{dx}\cot^{-1}x = \dfrac{-1}{1+x^2}$, $x\in\mathbb{R}$ $\quad [(\cot^{-1}u)' = \dfrac{-1}{1+u^2}u']$

(5) $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$ $\quad [(\sec^{-1}u)' = \dfrac{1}{|u|\sqrt{u^2-1}}u']$

(6) $\dfrac{d}{dx}\csc^{-1}x = \dfrac{-1}{|x|\sqrt{x^2-1}}$, $|x|>1$ $\quad [(\csc^{-1}u)' = \dfrac{-1}{|u|\sqrt{u^2-1}}u']$

Ex 5: Find y' if (a) $y = \sin^{-1}(3x)$ (b) $y = \dfrac{1}{\tan^{-1}x}$ (c) $y = e^x\sec^{-1}(x^2)$

3. Derivatives of hyperbolic functions

Theorem:

(1) $\dfrac{d}{dx}\sinh x = \cosh x$ $[\ (\sinh u)' = \cosh u \cdot u'\ ]$

(2) $\dfrac{d}{dx}\cosh x = \sinh x$ $[\ (\cosh u)' = \sinh u \cdot u'\ ]$

(3) $\dfrac{d}{dx}\tanh x = \text{sech}^2 x$ $[\ (\tanh u)' = \text{sech}^2 u \cdot u'\ ]$

(4) $\dfrac{d}{dx}\coth x = -\text{csch}^2 x$ $[\ (\coth u)' = -\text{csch}^2 u \cdot u'\ ]$

(5) $\dfrac{d}{dx}\text{sech}\, x = -\text{sech}\, x \cdot \tanh x$ $[\ (\text{sech}\, u)' = -\text{sech}\, u \cdot \tanh u \cdot u'\ ]$

(6) $\dfrac{d}{dx}\text{csch}\, x = -\text{csch}\, x \cdot \coth x$ $[\ (\text{csch}\, u)' = -\text{csch}\, u \cdot \coth u \cdot u'\ ]$

Ex 6: Find y' if (a) $y = \sinh 2x$ (b) $y = \cosh^2 3x$ (c) $y = \ln|\tanh x + \text{sech}\, x|$

4. Derivatives of inverse hyperbolic functions

Theorem:

(1) $\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{x^2+1}},\quad x\in\mathbb{R}$ $\qquad [\,(\sinh^{-1}u)'=\frac{1}{\sqrt{u^2+1}}\cdot u'\,]$

(2) $\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^2-1}},\quad x>1$ $\qquad [\,(\cosh^{-1}u)'=\frac{1}{\sqrt{u^2-1}}\cdot u'\,]$

(3) $\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2},\quad -1<x<1$ $\qquad [\,(\tanh^{-1}u)'=\frac{1}{1-u^2}\cdot u'\,]$

(4) $\frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^2},\quad x>1$ 或 $x<-1$ $\qquad [\,(\coth^{-1}u)'=\frac{1}{1-u^2}\cdot u'\,]$

(5) $\frac{d}{dx}\operatorname{sech}^{-1}x=\frac{-1}{x\sqrt{1-x^2}},\quad 0<x<1$ $\qquad [\,(\operatorname{sech}^{-1}u)'=\frac{-1}{u\sqrt{1-u^2}}\cdot u'\,]$

(6) $\frac{d}{dx}\operatorname{csch}^{-1}x=\frac{-1}{|x|\sqrt{1+x^2}},\quad x\in\mathbb{R}\setminus\{0\}$ $\qquad [\,(\operatorname{csch}^{-1}u)'=\frac{-1}{|u|\sqrt{1+u^2}}\cdot u'\,]$

Ex 7: Find y' if (a) $y=\sinh^{-1}2x$ (b) $y=\tanh^{-1}(\sqrt{3x^2-1})$

Ex 8: If $y=\cosh^{-1}(3x+y)$, find y'