3.5 Derivatives of exponential functions and logarithmic functions

1. Derivatives of exponential functions

Define: Euler number or Napier number

$$e = \lim_{n\to\infty}(1+\tfrac{1}{n})^n$$

$$= \lim_{x\to 0}(1+x)^{1/x}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!} = 2.718281828459045235360 28$$

Ex 1: Find $\lim_{n\to\infty}(1+\tfrac{3}{n})^n$

Ex 2: $\lim_{x\to 0}(1+2x)^{1/3x}$

Theorem: (1) $(e^x)' = e^x$ $\left[\!\left[(e^u)' = e^u \cdot u'\right]\!\right]$

(2) $(a^x)' = a^x \ln a$ $\left[\!\left[(a^u)' = a^u \ln a \cdot u'\right]\!\right]$

Ex 3: (a) $(e^{2x})'$

(b) $(e^{e^x})'$

(c) $(x^3 + 3^x)'$

(d) If $y = x^{2^3} + 3^{x^2} + 2^{3^x}$, find y'

(e) If $x + e^y = e^{(x+2y)} + 5$, find y'

2. **Derivatives of logarithmic functions**

Theorem: (1) $(\ln x)' = \frac{1}{x} \quad \forall x > 0$

(2) $(\ln |x|)' = \frac{1}{x} \quad \forall x \neq 0$

(3) $(\log_a |x|)' = \frac{1}{x \ln a}, \quad \forall x \neq 0$

Ex 4: (a) $\frac{d}{dx} \ln |x^2 - 1|$

(b) $\frac{d}{dx} \ln(\ln x)$

(c) $\frac{d}{dx} \log_2 x$

Ex 5: Show that $\frac{d}{dx} x^r = r x^{r-1}, \forall r \in \mathbb{R}$

Ex 6: If $xe^y + \ln(x^2 + y^2) = 10$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$

Ex 7: Differentiate $y = \frac{xe^{x^3+x}}{\sqrt{x^2+1}\sqrt[3]{x+3}}$

Theorem: (4)

$$(f(x)^{g(x)})' = f(x)^{g(x)}(g(x) \ln f(x))'$$
$$= f(x)^{g(x)}(g'(x) \ln f(x) + g(x)\frac{f'(x)}{f(x)})$$

Ex 8: If $y = (\ln x)^x$, find y'