3.4 Derivatives of implicit functions and inverse functions

1. Implicit Differentiation

Explicit function(顯函數): $y = f(x) = \dfrac{x^2+1}{x^2-1}$, $y = (x^3+1)^5$

Implicit function(隱函數): $(x^2-1)y = x^2 - 2x + 1$, $x^3y^5 - x = y^2 + 1$

Ex 1: $y^2 = x$, y is not a function of x that y is defined implicitly as a function of x.

Ex 2: $y = \sqrt{x}$ or $y = -\sqrt{x}$, y is as a function of x.

How to find $\dfrac{dy}{dx}$ for $F(x,y) = 0$?

Implicit differentiation:

Step1. Differentiate both sides of the equation with respect of x. (Using the Chain rule)

Step2. Solve the resulting equation for $\dfrac{dy}{dx}$.

Ex 3: If $y^2 = x$, find $\dfrac{dy}{dx}|_{(1,1)}$ and $\dfrac{dy}{dx}|_{(1,-1)}$.

Ex 4: Find an equation of the tangent to the curve $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.

Ex 5: Find y' if $\sqrt{x^2+y^2} - x^2 = 5$.

Ex 6: Find $\dfrac{dy}{dx}|_{(-1,-1)}$ if $\sqrt{\dfrac{y}{x}} + 3\sqrt{xy} = 4$.

2. Derivative of inverse functions

Theorem: $y = f(x) \Leftrightarrow x = f^{-1}(y)$

➔ $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ if $f'(x) \neq 0$

or $\dfrac{df^{-1}(y)}{dy} = \dfrac{1}{\dfrac{df(x)}{dx}}$

or $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}}$

Ex 7: If $f(x) = 2x^3 - 1$, find $(f^{-1})'(x)$

Ex 8: If $f(x) = x^5 + 2x^3 + 1$, find $(f^{-1})'(4)$

Ex 9: If $x^3 - y^3 = xy + 1$, find $\dfrac{dx}{dy}$

3. Differentiating with a parameter

Theorem: Derivatives to parametric curve

Let the parametric curve be $C: \begin{cases} x = x(t) \\ y = y(t) \end{cases}$. Then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ if $\dfrac{dx}{dt} \neq 0$.

Ex 10: Find an equation of the tangent line to the parameter curve $x = \dfrac{1}{t+1}, y = (t-1)^2$ at the point $(\frac{1}{3}, 1)$.