Chapter 3 Derivative
3.1 Derivative
1. Slope of a Tangent Line (切線斜率)

What is the slope of the tangent line to the graph $f(x)$ at the point $P(a, f(a))$?

Let $Q(a+h, f(a+h))$ and $Q \to P$

$\Rightarrow \overline{PQ} \to T$: the tangent line

$\Rightarrow m_{\overline{PQ}} \to m_T$: the slope of tangent line.

$$\therefore m_T = \lim_{Q \to P} m_{\overline{PQ}}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

2. Rates of Change(變率), Velocity(速度).

Ex 1: If a ball is thrown into the air with $v_0 = 50 m/s$, its height after t sec is given by $f(t) = 50t - 4.9t^2$.

(1) Find the average velocity over the time interval $[1, 1+h]$?

(2) What is the instantaneous velocity(瞬間速度) at $t = 1$?

Sol: (1) The average velocity

$$\overline{V} = \frac{f(1+h) - f(1)}{h} = 40.2 - 4.9h.$$

h	$\overline{V}$	h	$\overline{V}$
1	35.3	-1	45.1
0.1	39.71	-0.1	40.69
0.01	40.151	-0.01	40.249
0.001	40.1951	-0.001	40.2049
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0^+	40.2	0^-	40.2

(2) Def: $V(1) = 40.2 m/s$

i.e., $V(1) = \lim_{h \to 0} \overline{V} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 40.2 m/s$

$$V(a) = \lim_{h \to 0} \overline{V} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Def: Derivative

The derivative of f at a is given by

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

if the limit exists.

Ex 2: If $f(x)=x(x+1)(x+2)$, find $f'(-1)$.

Def: Derivative of a function

The derivative of f with respect to x is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Ex 3: If $f(x)=\sqrt{x+2}$, find $f'(x)$.

Concept: (1) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

(2) $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$=f'(x)\big|_{x=a}$$

(3) $f'(x)=\frac{df(x)}{dx}=D_x f(x)$,

$$f'(a)=f'(x)\big|_{x=a}=\frac{df}{dx}\bigg|_{x=a}=D_x f(x)\big|_{x=a}$$

(4) $f'(a) = f(x)$ 在 a 點的第一階導數(the first derivative)
$= f(x)$ 在 a 點的切線斜率(slope of the tangent line)
$= f(x)$ 在 a 點的變率(rates of change)
$= f(x)$ 在 a 點的邊際值(marginal)

Ex 4: If $f(x) = x^2$, find (1) $f'(x)$ (2) $f'(2)$.

Ex 5: Let $f(x) = \dfrac{1}{x}$.

(1) Compute $f'(x)$
(2) Find the slope of the tangent line T to the graph of f at $x = 1$.
(3) Find an equation of the tangent line to the curve at $x = 1$.
(4) What is the rate of change of f at the point?

Def: (1) Derivative from the right (右導數):

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

(2) Derivative from the left (左導數):

$$f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Theorem: If $f'(a)$ exists $\Leftrightarrow$ $f'_+(a) = f'_-(a) = f'(a)$

So, if $f'_+(a) \neq f'_-(a)$ $\Rightarrow$ $f'(a)$ dose not exist.

Ex 6: If $f(x) = |x|$, find $f'(0)$.

Def: A function f is differentiable(可微分) at $x = a$ if $f'(a)$ exists.

Theorem: Differentiable implies continuity

If f is differentiable at $x = a$ $\Rightarrow$ f is continuous at $x = a$.

Concept:

(1) The converse proposition is wrong.

Ex 7: $f(x) = x^{1/3}$ or $f(x) = |x|$.

(2) If f is not continuous at $x = a$ $\Rightarrow$ $f'(a)$ dose not exist.

Ex 8: If $f(x) = \begin{cases} 2x - 3, x \geq 1 \\ 2 - x \ , x < 1 \end{cases}$, find $f'(1)$