2.1 Limits(極限)

Ex 1: How to find the area of unit circle?

Ex 2: 一尺之棰，日去其半，萬世不竭。

Ex 3: Evaluate $\lim_{x\to 2} x^3$

Def: If $f(x)$ becomes arbitrarily close to a single number L as x approaches a from either side, the limit of $f(x)$, as x approaches a, is L. This limit is written by

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to a.$$

["$\to$" approach] [K. Weierstrass (1815-1897)]

Ex 4: If $f(x) = \dfrac{x-1}{x^2-1}, x \neq 1$ then $f(x) \to ?$ as $x \to 1$.

$x<1$	$f(x)$	$x>1$	$f(x)$	$\lvert x-1\rvert$	$\lvert f(x)-0.5\rvert$
0.5	0.6667	1.5	0.4000	0.5	<0.25
0.9	0.5263	1.1	0.4761	0.1	<0.05
0.99	0.5025	1.01	0.4975	0.01	<0.005
0.999	0.5002	1.001	0.4997	0.001	<0.0005
0.9999	0.5000	1.0001	0.4999	0.0001	<0.00005
				$<\delta$ (delta)	$<\frac{\delta}{2}=\varepsilon$ (epsilon)

Thus, $f(x) \to \frac{1}{2}$ as $x \to 1$ or $\lim_{x\to 1} f(x) = \frac{1}{2}$.

Def: (limit)

$$\lim_{x\to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni \text{ if } 0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

Ex 5: Prove that $\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$

Ex 6: Let

$$g(x) = \begin{cases} x+2, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

Evaluate $\lim_{x \to 1} g(x)$

Theorem: If the limit of a function exists, it is unique.

i.e, If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M \Rightarrow L = M$

Ex 7: Find $\lim_{x \to 0} f(x)$ if $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

Ex 8: Discuss the existence of the limit

(1) $\lim_{x \to 0} \frac{1}{x^2}$

(2) $\lim_{x \to 0} \sin \frac{1}{x}$

Theorem: (properties of limits)

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

Then (1) $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x) = cL$

(2) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

(3) $\lim_{x\to a}(f(x)\cdot g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x)) = LM$

(4) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}, M \neq 0$

Ex 9: (1) $\lim_{x\to 3} 2x^3\sqrt{x^2+7}$ (2) $\lim_{x\to 2}\dfrac{2x^2+1}{x+1}$

Indeterminate form:

Ex 10: $\lim_{x\to 2}\dfrac{4(x^2-4)}{x-2}$

Ex 11: $\lim_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}$

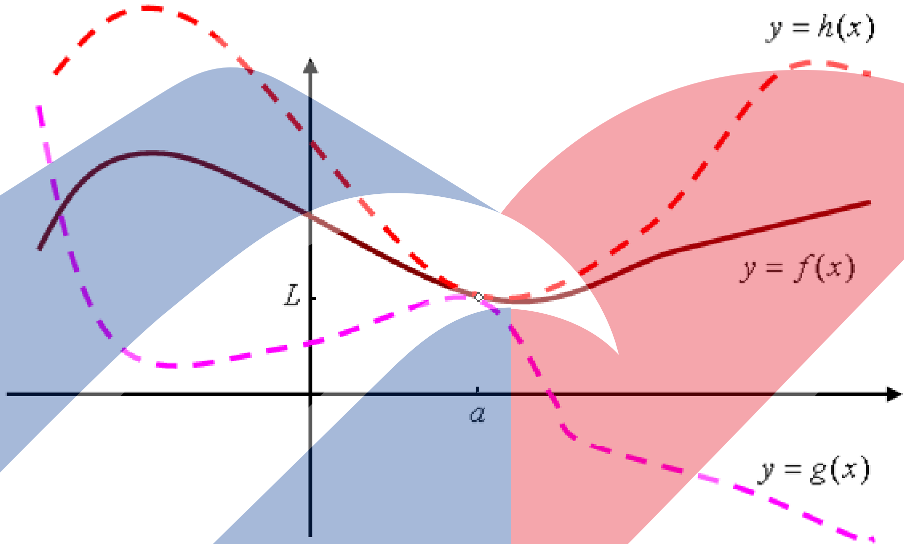
Ex 12: If $f(x) = \dfrac{1}{x}$, find $\lim_{x\to 3}\dfrac{1}{x-3}(f(x+1)-f(4))$

The squeeze theorem (夾擠定理)

If $\exists \delta > 0, \ni \quad g(x) \le f(x) \le h(x) \quad \forall x \in (a-\delta, a) \cup (a, a+\delta)$ and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} f(x) = L$



Ex 13: Show that $\lim_{x \to a} f(x) = 0 \Leftrightarrow \lim_{x \to a} |f(x)| = 0$

Ex 14: Find $\lim_{x \to 0} x \sin \frac{1}{x}$

Ex 15: Find $\lim_{x \to 0} x^2 \sin \frac{1}{x}$