1.6 Hyperbolic functions and inverse hyperbolic functions

1. Hyperbolic functions

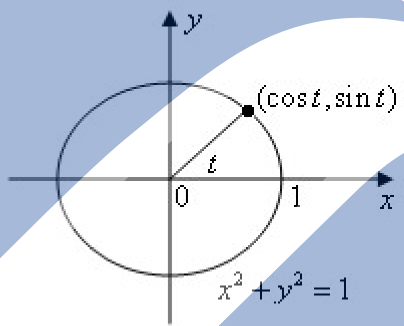
Def: hyperbolic functions（雙曲函數）

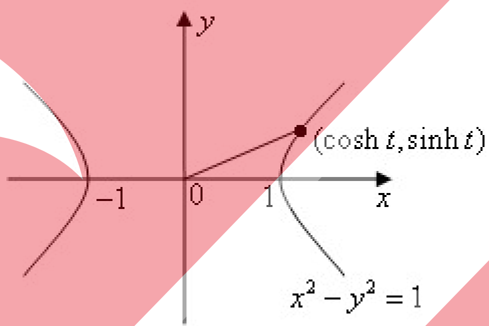
$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Concept:





Formulae:

(1) $\sinh(-x) = -\sinh x$

(2) $\cosh(-x) = \cosh x$

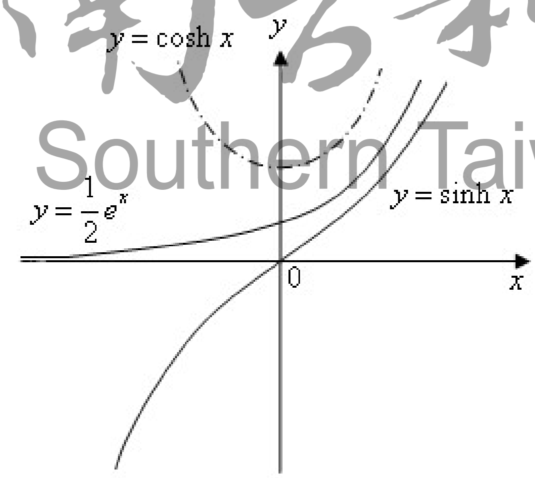
(3) $\cosh^2 x - \sinh^2 x = 1$

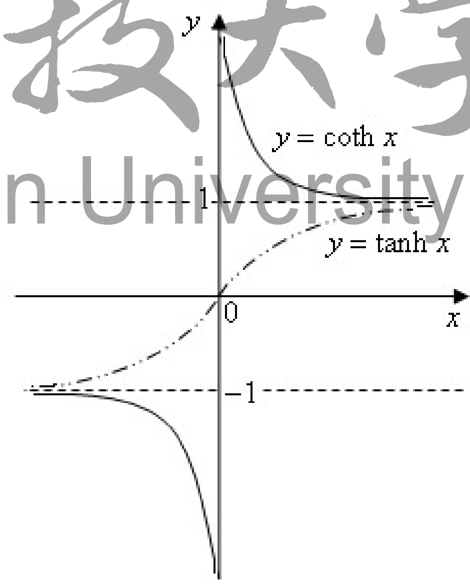
(4) $1 - \tanh^2 x = \operatorname{sech}^2 x$

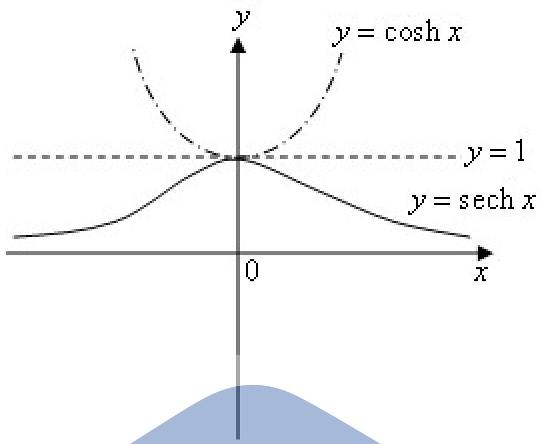
(5) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

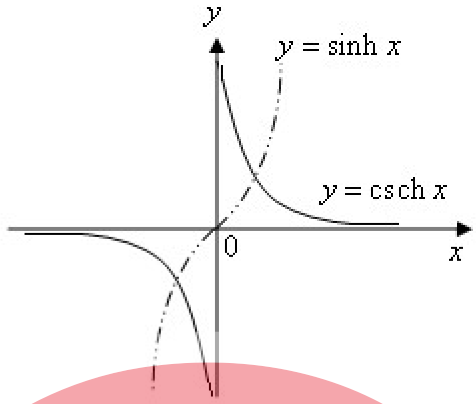
(6) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Graphs:









2. Inverse hyperbolic functions

Def: $y=\sinh^{-1}x \Leftrightarrow x=\sinh y$

$y=\cosh^{-1}x \Leftrightarrow x=\cosh y$ and $y\ge 0,\ \ x\ge 1$

$y=\tanh^{-1}x \Leftrightarrow x=\tanh y$

Theorem:

(1) $\sinh^{-1}x=\ln(x+\sqrt{x^2+1}),\quad x\in\mathbb{R}$

(2) $\cosh^{-1}x=\ln(x+\sqrt{x^2-1}),\quad x\ge 1$

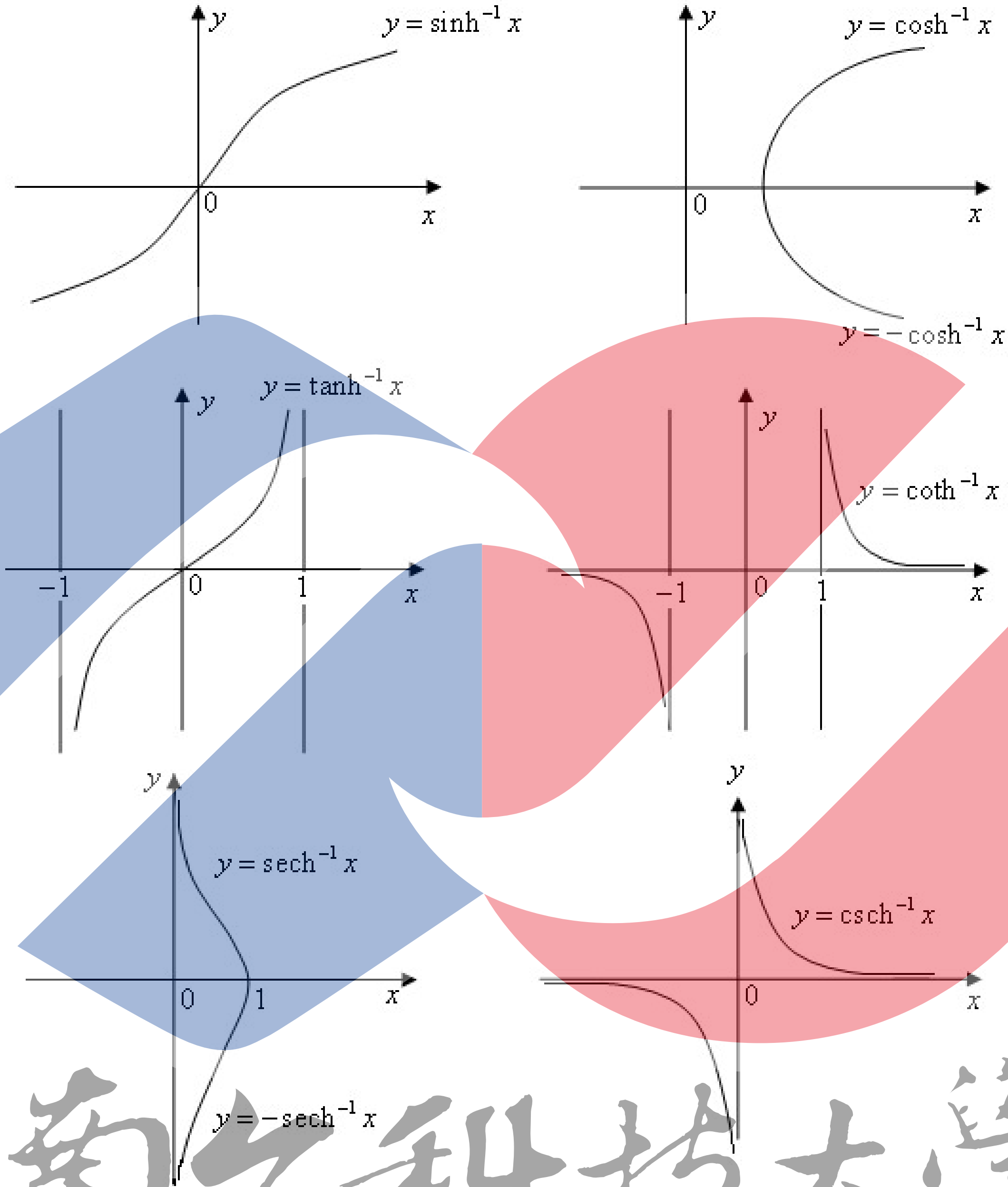
(3) $\tanh^{-1}x=\dfrac{1}{2}\ln(\dfrac{1+x}{1-x}),\quad -1<x<1$

(4) $\coth^{-1}x=\dfrac{1}{2}\ln(\dfrac{x+1}{x-1}),\quad |x|>1$

(5) $\operatorname{sec}h^{-1}x=\ln(\dfrac{1+\sqrt{1-x^2}}{x}),\quad 0<x\le 1$

(6) $\operatorname{csc}h^{-1}x=\ln(\dfrac{1\pm\sqrt{1+x^2}}{x}),\quad x\ne 0$

Graph：


y = sinh⁻¹ x
y
0
x
y = cosh⁻¹ x
y = −cosh⁻¹ x
y = tanh⁻¹ x
−1
1
y = coth⁻¹ x
y = sech⁻¹ x
y = −sech⁻¹ x
y = csch⁻¹ x