1.4 Exponential functions and logarithmic functions

1. Exponential functions

Def: Euler number (歐拉數) or Napier number (納皮爾數)

$$e = \lim_{n\to\infty}(1+\tfrac{1}{n})^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}$$
$$= 2.718281828459045235360\,28$$

Def: If $a > 0$ is the base and x is exponent variable, then the exponential function is defined by

$$y = a^x$$

If $a = e$, then

$$y = e^x$$

is called the natural exponential function (自然指數函數).

Laws of exponents:

(1) $a^x \cdot a^y = a^{x+y}$ $\left[\!\left[e^x \cdot e^y = e^{x+y} \right]\!\right]$

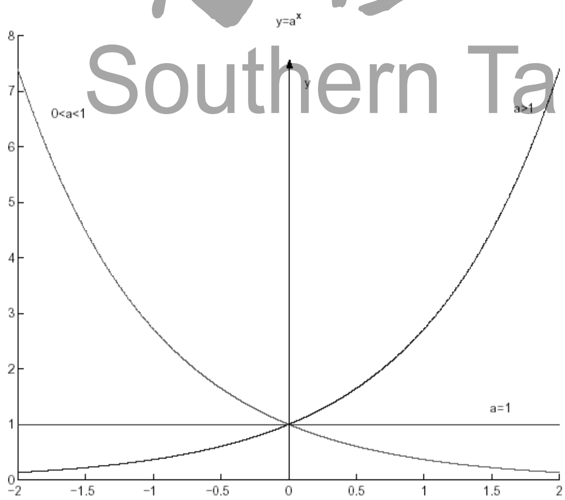
(2) $\dfrac{a^x}{a^y} = a^{x-y}$ $\left[\!\left[\dfrac{e^x}{e^y} = e^{x-y} \right]\!\right]$

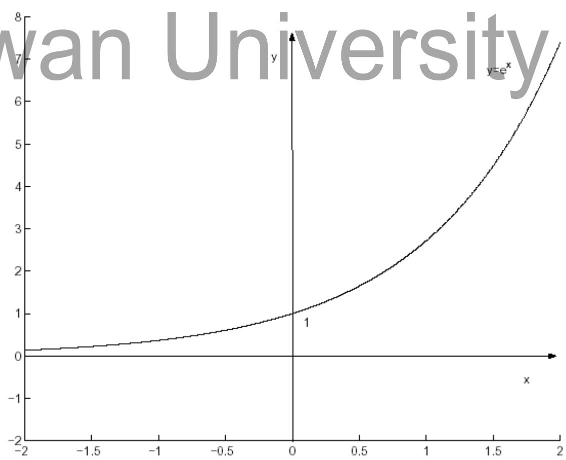
(3) $(a^x)^y = a^{xy}$ $\left[\!\left[(e^x)^y = e^{xy} \right]\!\right]$

(4) $(ab)^x = a^x b^x$

Ex 1: Solve the equation $e^{2x+1} / e^3 = e^{x-1}$.

Graphs:





Ex 2: Sketch the graph of $y = e^{-x}$.

Def: Exponential Growth, Exponential Decay

The function $y = ka^x$, $k > 0$ is a model for exponential growth if $a > 1$, and a model for exponential decay if $0 < a < 1$.

2. Logarithmic functions

Def: If $a > 0, a \neq 1$ is the base and $x > 0$ is variable, then the logarithmic function (對數函數) is defined by

$$y = \log_a x.$$

If $a = e$, then

$$y = \log_e x \equiv \ln x$$

is called the natural Logarithmic function (自然對數函數).

Laws of Logarithms:

(1) $\log_a(xy) = \log_a x + \log_a y$ $\quad [\![\ln(xy) = \ln x + \ln y]\!]$

(2) $\log_a(\frac{y}{x}) = \log_a y - \log_a x$ $\quad [\![\ln(\frac{y}{x}) = \ln y - \ln x]\!]$

(3) $\log_a x^y = y \log_a x$ $\quad [\![\ln x^y = y \ln x]\!]$

(4) $\log_x y = \dfrac{\log_a y}{\log_a x} = \dfrac{\ln y}{\ln x}$

Concept: $y = a^x$ and $y = \log_a x$ are inverse functions to each other.

$$8 = 2^3 \Leftrightarrow 3 = \log_2 8.$$

So,

$$y = a^x \Leftrightarrow x = \log_a y \quad [\![y = e^x \Leftrightarrow x = \ln y]\!]$$

(1) $a^{\log_a y} = y \quad \forall \ y > 0$ $\quad [\![e^{\ln y} = y]\!]$

(2) $\log_a a^x = x \quad \forall \ x \in R$ $\quad [\![\ln e^x = x]\!]$

Ex 3: (1) $\ln e^3 = ?$

(2) $\ln e = ?$

(3) $e^{\ln 3} = ?$

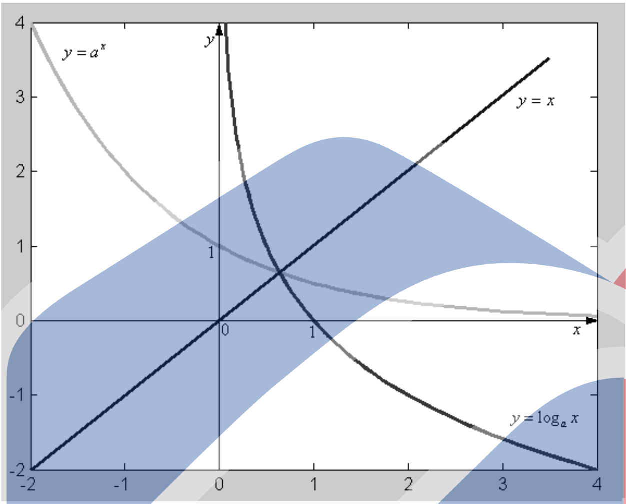
(4) $\ln 1 = ?$

Ex 4: If $2e^{x+2} = 5$, find x.

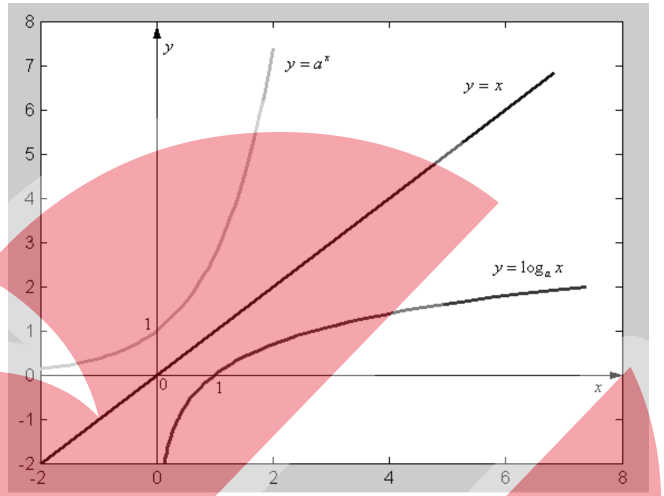
Ex 5: Solve the equation $\ln|5-x| = 7$.

Graphs:

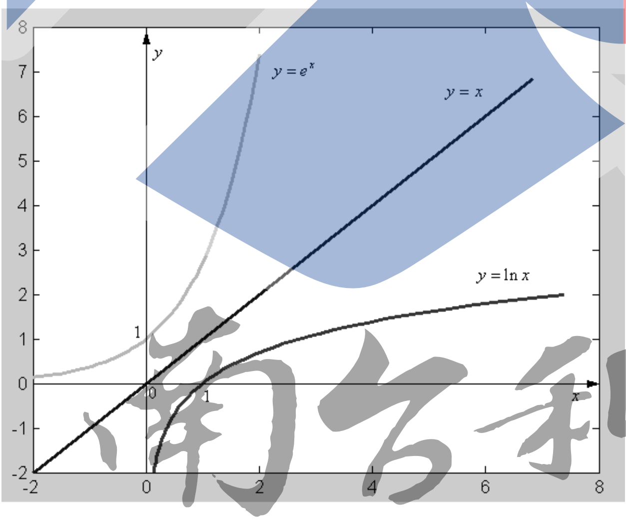
(1) $0 < a < 1$



(2) $a > 1$



(3) $a = e$



Ex 6: Sketch the graph of $y = \ln(x-2) - 1$