

# 1. Preliminaries

## 1.1 functions

### 1. Real system(實數系):

$$\mathbb{R}(\text{real}) = \begin{cases} \mathbb{Q}(\text{rational}) = \begin{cases} \mathbb{Z}(\text{integer}) = \begin{cases} \mathbb{N} = \mathbb{Z}^+ (\text{positive integer or natural number}) \\ 0 \\ \mathbb{Z}^- (\text{negative integer}) \end{cases} \\ \mathbb{Q} \setminus \mathbb{Z} \\ \mathbb{R} \setminus \mathbb{Q} = \mathbb{Q}^c (\text{irrational}) \end{cases} \end{cases}$$

### 2. Intervals(區間):

Define: finite intervals(有限區間)

(1) Open interval(開區間)

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

(2) Closed interval(閉區間)

$$[a, b] = \{x \mid a \leq x \leq b\}$$

(3) Half-open interval(半開區間)

$$[a, b) = \{x \mid a \leq x < b\}, (a, b] = \{x \mid a < x \leq b\}$$

Def: infinite intervals(無限區間)

(1)  $(-\infty, b] = \{x \mid x \leq b\}$ ,  $(-\infty, b) = \{x \mid x < b\}$

(2)  $(a, \infty) = \{x \mid x > a\}$ ,  $[a, \infty) = \{x \mid x \geq a\}$

(3)  $(-\infty, \infty) = \{x \mid x \in \mathbb{R}\} = \mathbb{R}$

### 3. Union(聯集) and intersection(交集)

Define: (1)  $A \cup B = \{x \mid x \in A \vee x \in B\}$  (the union of  $A$  and  $B$ )

(2)  $A \cap B = \{x \mid x \in A \wedge x \in B\}$  (the intersection of  $A$  and  $B$ )

Ex1: If  $A = (-2, 3)$ ,  $B = (0, 5)$ ,  $C = [2, \infty)$ , find  $A \cup B$ ,  $A \cap B$ ,  $A \cap B \cap C$ ,  $A \cup B \cup C$

### 4. Functions(函數):

Def: If  $f : A \rightarrow B$  is a function.  $\Leftrightarrow \forall x \in A, \exists! y \in B \ni f(x) = y$

The set  $A$  is called the domain (定義域) of  $f$ ,  $\llbracket A \equiv D_f \rrbracket$ .

$B$  is called the codomain (對應域).

$f(A) \equiv \{f(x) \mid \forall x \in A\}$  is the range or value (像或值域) of  $f$  in  $A$ ,  $\llbracket f(A) \equiv R_f \rrbracket$

Ex2: Find the domain of  $f(x)$  if (1)  $f(x) = x^2 + 3$ , (2)  $f(x) = \sqrt{x-1}$ , (3)  $f(x) = \frac{1}{x^2-4}$

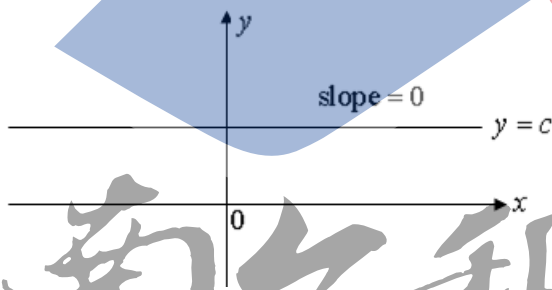
Ex3: If  $f(x) = \frac{1}{x+1}$ . Compute: (1)  $f(1)$  (2)  $f(-2)$  (3)  $f(a)$  (4)  $f(a+h)$   
 (5)  $\frac{f(a+h) - f(a)}{h}$ .

5. Polynomial functions (多項式函數):

Def:  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_i \in \mathbb{R}, a_n \neq 0$  is a polynomial function of degree  $n$ .

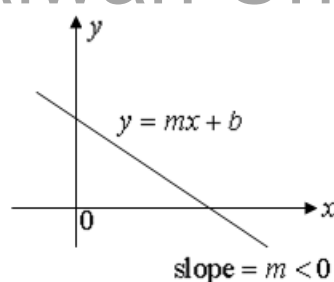
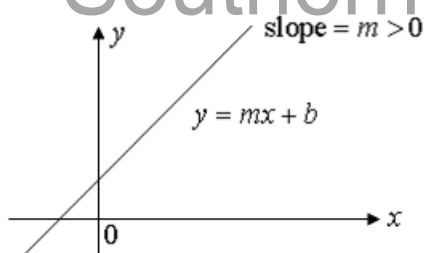
(1) Constant function (常數函數):

$$f(x) = c, c \in \mathbb{R} \quad \llbracket \text{slope} = 0 \rrbracket$$



(2) Linear function (線性函數):

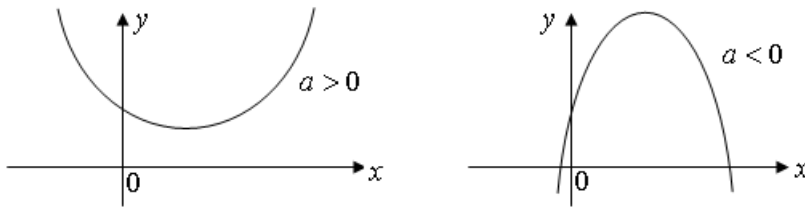
$$f(x) = mx + b, m \neq 0 \quad \llbracket \text{slope} = m \rrbracket$$



(3) Quadratic function (二次函數):

$$f(x) = ax^2 + bx + c, a \neq 0$$

Concept: The roots of  $f(x)=0$  are  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .



(4) Cubic function(三次函數):

$$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$$

Ex4:  $f(x) = x^3$

6. Power functions(冪函數):

$$f(x) = x^\alpha, \alpha \in \mathbb{R}$$

Ex5:  $f(x) = \sqrt{x}$

Ex6:  $f(x) = \frac{1}{x}$

7. Rational functions(有理函數):

$$f(x) = \frac{q(x)}{p(x)}, p(x), q(x) \in \text{polynomials}$$

Concept:  $D_f = \{x \mid p(x) \neq 0\}$

Ex7:  $f(x) = \frac{x^3 + 1}{x}$

8. Algebraic functions(代數函數):

Ex8:  $f(x) = \sqrt[3]{x + \sqrt{x}} - x^2$

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## 9. Composite function(合成函數)

Def:

$$(f \circ g)(x) \equiv f(g(x)) \quad (\text{read "f circle g"})$$

$$D_{f \circ g} = \{x \mid x \in D_g \text{ and } g(x) \in D_f\}$$

Ex9: Let  $f(x) = x^2 - 1$  and  $g(x) = \sqrt{x} + 1$ , find  $f \circ g$  and  $g \circ f$ 

## 10. Special functions:

(1) Conditional (piecewise) function (條件函數):

$$\text{Ex10: } f(x) = \begin{cases} x-1, & x \leq 1 \\ x^2, & x > 1 \end{cases}$$

$$\text{Ex11: } f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

(2) Absolute value function(絕對值函數):

$$\text{Ex12: } f(x) = \frac{|x-1|}{x-1}$$

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(3) Gauss function(高斯函數), or the greatest integer function(最大整數函數):

Def:  $[x] \equiv$  the largest integer that is less than or equal to  $x$ .

$$\text{i.e., } n \leq x < n+1 \Leftrightarrow [x] = n, \quad \forall n \in \mathbb{Z}$$

Ex13:  $[1.1] = 1, [1.99] = 1, [2.09] = 2, [\pi] = 3, [2] = 2, [-1.1] = -2, [-\pi] = -4$

Ex14:  $f(x) = [x]$

Ex15:  $f(x) = [x^2]$

## 11. Parametric Equations

Def: Parametric equation:  $\begin{cases} x = f(t) \\ y = g(t) \end{cases}, t \in I$

Ex16:  $x = \sqrt{t}, y = t, t \geq 0$

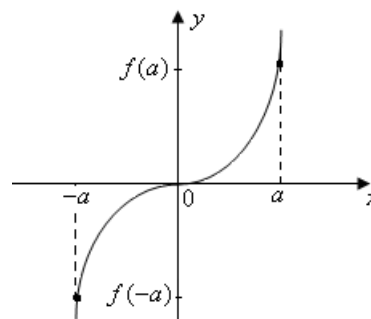
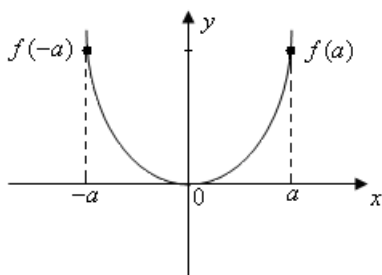
Ex17:  $x = a \cos t, y = a \sin t, t \in [0, 2\pi]$

## 12. Even function, Odd function

Def: (1)  $f$  is an even function (偶函數)  $\Leftrightarrow f(-x) = f(x), \forall x \in D_f$

$\Leftrightarrow$  the graph of  $f$  is symmetric about the y-axis

Ex:  $f(x) = x^2, x^4, \cos x, \frac{x^2}{1+x^4}$

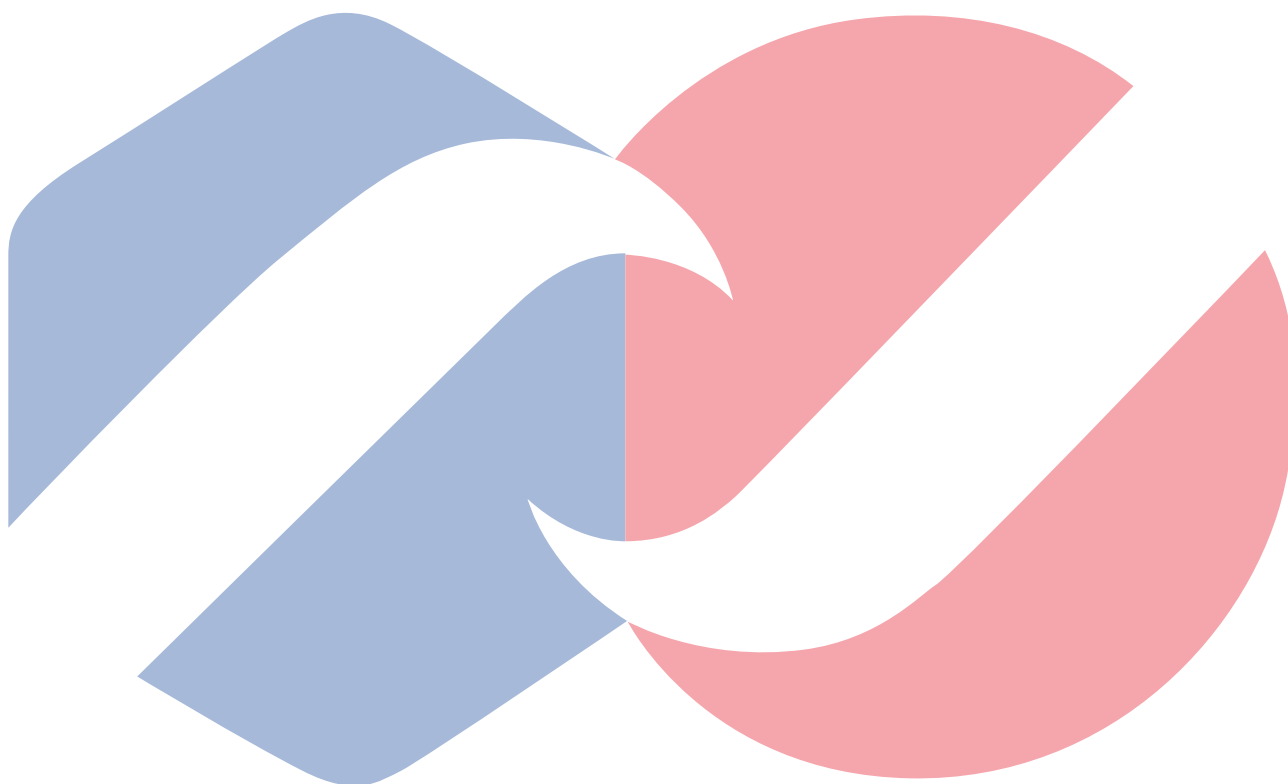


(2)  $f$  is an odd function(奇函數)  $\Leftrightarrow f(-x) = -f(x), \forall x \in D_f$

$\Leftrightarrow$  the graph of  $f$  is symmetric about the origin

Ex:  $f(x) = x, x^3, \sin x, x^2 \sin x$

Ex18: Show that  $f(x) = \sin(x^3)$  is an odd function.



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