

## Chapter 4 Ordinary Differential Equations of Higher Order

## I. Linear differential equations of higher order

$$\text{Form: } p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + p_{n-2}(x)y^{(n-2)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = r(x) \quad (4.1)$$

$$\text{or} \quad \mathbf{L}y(x) = r(x)$$

$$\text{where } L = p_n \frac{d^n}{dx^n} + p_{n-1} \frac{d^{n-1}}{dx^{n-1}} + p_{n-2} \frac{d^{n-2}}{dx^{n-2}} + \cdots + p_1 \frac{d}{dx} + p_0$$

## 1. Linear dependence

Definition: A linear combination of  $n$  functions  $y_1(x), y_2(x), \dots, y_n(x)$ , is an expression of the form

$$k_1y_1(x) + k_2y_2(x) + \cdots + k_ny_n(x)$$

where  $k_i$ 's are constants.

**Definition:** If the equation

$$k_1y_1(x) + k_2y_2(x) + \cdots + k_ny_n(x) = 0 \quad (4.2)$$

holds on an interval I for some  $k_1, k_2, \dots, k_n$  not all zero, the function  $y_1(x), y_2(x), \dots, y_n(x)$  are said to be linearly dependent. If  $y_1(x), y_2(x), \dots, y_n(x)$  are linearly dependent, there exists at least one constant, say  $k_i$ , not zero, and the corresponding function  $y_i(x)$  can be expressed as a linear combination of the other functions, i.e.,

$$y_i(x) = -\frac{1}{k_i} [k_1 y_1(x) + k_2 y_2(x) + \cdots + k_{i-1} y_{i-1}(x) + k_{i+1} y_{i+1}(x) + \cdots + k_n y_n(x)] \quad (4.3)$$

Definition: If (4.2) holds on interval I only for  $k_1=k_2=\dots=k_n=0$ , then  $y_1(x), y_2(x), \dots, y_n(x)$  are said to be linearly independent.

Assume that each of a set of  $n$  functions  $y_1(x), y_2(x), \dots, y_n(x)$  possesses  $n$  finite derivatives at all points of an interval  $I$ . Then

$$\begin{aligned} k_1 \frac{dy_1}{dx} + k_2 \frac{dy_2}{dx} + \cdots + k_n \frac{dy_n}{dx} &= 0 \\ k_1 \frac{d^2y_1}{dx^2} + k_2 \frac{d^2y_2}{dx^2} + \cdots + k_n \frac{d^2y_n}{dx^2} &= 0 \end{aligned} \quad (4.4)$$

$$k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + k_n \frac{d^{n-1} y_n}{dx^{n-1}} = 0$$

If the functions  $y_1(x), y_2(x), \dots, y_n(x)$  are linearly dependent, i.e., there exists nontrivial solution of  $k_i$ 's, the determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \quad (4.5)$$

vanishes identically over I. This determinant is called the Wronskian of the functions  $y_1(x), y_2(x), \dots, y_n(x)$ . If the Wronskian of  $y_1(x), y_2(x), \dots, y_n(x)$  is not identically zero over I., then the functions  $y_1(x), y_2(x), \dots, y_n(x)$  are linearly independent.

[Note] The vanishing of the Wronskian is necessary but not sufficient condition for linear independence of a set of functions.

**Ex. 1**

Show that  $e^x, e^{2x}$  are linearly independent on all interval.

$$\text{Solution: } W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x e^{2x} \neq 0$$

**Ex. 2**

Show that  $x, x^2$  are linearly independent on  $\{x | x \in \mathbb{R}, x \neq 0\}$ .

$$\text{Solution: } W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0 \text{ if } x \neq 0.$$

Principle of superposition: If  $y_1(x), y_2(x)$  are the solutions of the homogeneous linear differential equation

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0 \quad (4.6)$$

then  $k_1y_1(x) + k_2y_2(x)$  is also a solution of (4.6), where  $k_1, k_2$  are arbitrary constants.

An  $n$ th order linear differential equation (4.6) has  $n$  linearly independent solutions  $y_1(x), y_2(x), \dots, y_n(x)$ , and its general solution can be expressed as

$$y_h(x) = k_1y_1(x) + k_2y_2(x) + \cdots + k_ny_n(x)$$

where  $k_i$ 's are arbitrary constants, and  $y_h(x)$  is called homogeneous solution or complementary solution of (4.1). Now suppose that one particular solution of (4.1), say  $y_p(x)$ , then the complete solution of (4.1) is

$$y(x) = y_h(x) + y_p(x)$$

## 2. Linear differential equation with constant coefficients

Form:  $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = r(x)$  (4.7)

or  $Ly = r(x)$

where  $a_0, a_1, \dots, a_n$  are constants.

(1) Homogeneous solution:

Let  $y_h(x) = e^{\lambda x}$ , substitute into the homogeneous equation of (4.7), i.e.,  $r(x) = 0$ , we have

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0$$

Since  $e^{\lambda x} \neq 0$ , we get

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

Equation (4.8) is called the characteristic equation of the homogeneous equation (4.7)

Case 1: If (4.8) has  $n$  distinct real roots, say  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then

$$y_h(x) = k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x} + \cdots + k_n e^{\lambda_n x}$$

Case 2: If (4.8) has a real double root, say  $\lambda_1 = \lambda_2$ ,

$$\because L e^{\lambda x} = a_n (\lambda - \lambda_1)^2 (\lambda - \lambda_3) (\lambda - \lambda_4) \cdots (\lambda - \lambda_n) e^{\lambda x}$$

$$\therefore L e^{\lambda x} \Big|_{\lambda=\lambda_1} = 0, \quad e^{\lambda_1 x} \text{ is a solution of } Ly=0.$$

and

$$\begin{aligned} \frac{\partial}{\partial \lambda} (L e^{\lambda x}) &= 2(\lambda - \lambda_1)(\lambda - \lambda_3)(\lambda - \lambda_4) \cdots (\lambda - \lambda_n) e^{\lambda x} \\ &\quad + (\lambda - \lambda_1)^2 (\lambda - \lambda_4)(\lambda - \lambda_5) \cdots (\lambda - \lambda_n) e^{\lambda x} \\ &\quad + (\lambda - \lambda_1)^2 (\lambda - \lambda_3)(\lambda - \lambda_5) \cdots (\lambda - \lambda_n) e^{\lambda x} \\ &\quad + \cdots \\ &\quad + (\lambda - \lambda_1)^2 (\lambda - \lambda_3)(\lambda - \lambda_4) \cdots (\lambda - \lambda_{n-1}) e^{\lambda x} \end{aligned}$$

$$\therefore \frac{\partial}{\partial \lambda} (L e^{\lambda x}) \Big|_{\lambda=\lambda_1} = 0$$

$$\text{but } \frac{\partial}{\partial \lambda} (L e^{\lambda x}) \Big|_{\lambda=\lambda_1} = \left( \frac{\partial e^{\lambda x}}{\partial \lambda} \right) \Bigg|_{\lambda=\lambda_1} = L(x e^{\lambda_1 x}) = 0$$

$x e^{\lambda_1 x}$  is also a solution of  $Ly=0$ . By a simple extension of this argument, it can be shown that the part of the homogeneous solution corresponding to an  $m$ -fold root  $\lambda_1$  is

$$(k_1 + k_2 x + k_3 x^2 + \cdots + k_m x^{m-1}) e^{\lambda_1 x}$$

**Ex. 3**

$$y''' - 2y'' - y' + 2y = 0.$$

Solution: The characteristic equation is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \Rightarrow (\lambda + 1)(\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = -1, 1, 2$$

The general solution is  $y = k_1 e^{-x} + k_2 e^x + k_3 e^{2x}$

**Ex. 4**

$$y^{(5)} - 3y^{(4)} + 3y^{(3)} - y'' = 0.$$

Solution: The characteristic equation is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 1)^3 = 0 \Rightarrow \lambda = 0, 0, 1, 1, 1$$

The general solution is  $y = k_1 + k_2 x + (k_3 + k_4 x + k_5 x^2) e^x$

Case 3: If (4.8) has simple complex root, say  $\lambda_1 = \alpha \pm i\beta$ , then solution

$$\begin{aligned} k_1 e^{(\alpha+i\beta)x} + k_2 e^{(\alpha-i\beta)x} &= k_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + k_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) = \\ &e^{\alpha x} [(k_1 + k_2) \cos \beta x + i(k_1 - k_2) \sin \beta x] = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) \end{aligned}$$

where  $C_1 = k_1 + k_2$ ,  $C_2 = i(k_1 - k_2)$

Hence the solution can be expressed as  $e^{\alpha x} (k_1 \cos \beta x + k_2 \sin \beta x)$

Case 4: If (4.8) has complex multiple roots, the solution is

$$e^{\alpha x} [(k_1 + k_2 x + \dots + k_m x^{m-1}) \cos \beta x + (k_{m+1} + k_{m+2} x + \dots + k_{2m} x^{m-1}) \sin \beta x]$$

**Ex. 5**

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$$y''' - 2y'' + 2y' = 0, y(0) = 0.5, y'(0) = -1, y''(0) = 2.$$

Solution: The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda^2 - 2\lambda + 2) = 0 \Rightarrow \lambda = 0, 1 \pm i$$

The general solution is  $y(x) = k_1 + e^x (k_2 \cos x + k_3 \sin x)$

$$y(0) = 0.5 \Rightarrow 0.5 = k_1 + k_2$$

$$y'(0) = -1 \Rightarrow -1 = k_2 + k_3 \Rightarrow k_1 = 2.5, k_2 = -2, k_3 = 1$$

$$y''(0) = 2 \Rightarrow 2 = 2k_3$$

$$\therefore y = 2.5 + e^x (-2 \cos x + \sin x)$$

**Ex. 6**

$$y^{(7)} + 18y^{(5)} + 81y^{(3)} = 0.$$

Solution: The characteristic equation is

$$\lambda^7 + 18\lambda^5 + 81\lambda^3 = 0 \Rightarrow \lambda^3(\lambda^2 + 9)^2 = 0 \Rightarrow \lambda = 0, 0, 0, \pm 3i, \pm 3i$$

The general solution is  $y(x) = k_1 + k_2x + k_3x^2 + (k_4 + k_5x)\cos 3x + (k_6 + k_7x)\sin 3x$

**(2) Particular solution**

a. Method of undetermined coefficients:

| Term in $r(x)$ | Choice for $y_p(x)$                               |
|----------------|---|
| $x^n$          | $c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$ |
| $e^{px}$       | $c e^{px}$  |
| $\cos qx$      | $A \cos qx + B \sin qx$                           |
| $\sin qx$      | $A \cos qx + B \sin qx$                           |

**Ex. 7**

$$y^{(4)} - y = 4.5e^{-2x}.$$

Solution: The characteristic equation is

$$\lambda^4 - 1 = 0 \Rightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0 \Rightarrow \lambda = \pm 1, \pm i$$

$$y_h(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{Let } y_p(x) = A e^{-2x} \Rightarrow y_p^{(4)}(x) = 16A e^{-2x}$$

$$y_p^{(4)} - y_p = 4.5e^{-2x} \Rightarrow (16A - A)e^{-2x} = 4.5e^{-2x} \Rightarrow A = 0.3$$

$$\text{The complete solution is } y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + 0.3e^{-2x}$$

**Ex. 8**

$$y''' - 3y'' + 3y' - y = 30e^x.$$

Solution: The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow (\lambda - 1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$$

$$y_h(x) = (k_1 + k_2x + k_3x^2)e^x$$

$$\text{Let } y_p(x) = Ax^3e^x \Rightarrow y_p'(x) = A(x^3 + 3x^2)e^x, y_p''(x) = A(x^3 + 6x^2 + 6x)e^x,$$

$$y_p'''(x) = A(x^3 + 9x^2 + 18x + 6)e^x$$

Substituting into the problem, we have

$$A[(x^3 + 9x^2 + 18x + 6) - 3(x^3 + 6x^2 + 6x) + 3(x^3 + 3x^2) - x^3]e^x = 30e^x \Rightarrow A = 5$$

$$\text{The complete solution is } y(x) = y_h(x) + y_p(x) = (k_1 + k_2x + k_3x^2 + 5x^3)e^x$$

**Ex. 9**

$$y'' - 2y' + y = x^2e^x.$$

Solution: The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1$$

$$y_h(x) = (k_1 + k_2x)e^x$$

$$\text{Let } y_p(x) = (Ax^4 + Bx^3 + Cx^2)e^x \Rightarrow y_p'(x) = [Ax^4 + (4A + B)x^3 + (3B + C)x^2 + 2Cx]e^x,$$

$$y_p''(x) = [Ax^4 + (8A + B)x^3 + (12A + 6B + C)x^2 + (6B + 4C)x + 2C]e^x,$$

Substituting into the problem, we have

$$[(A - 2A + A)x^4 + (8A + B - 8A - 2B + B)x^3 + (12A + 6B + C - 6B - 2C + C)x^2 + (6B + 4C - 4C)x + 2C]e^x = x^2e^x \Rightarrow A = 1/12, B = 0, C = 0$$

$$\text{The complete solution is } y(x) = y_h(x) + y_p(x) = (k_1 + k_2x + 1/12x^4)e^x$$

**Ex. 10**

$$y'' + 4y = x^2 + \cos x.$$

Solution: The characteristic equation is

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y_h(x) = k_1 \cos 2x + k_2 \sin 2x$$

$$\text{Let } y_p(x) = (Ax^2 + Bx + C) + (D \cos x + E \sin x)$$

$$y_p'(x) = 2Ax + B - D \sin x + E \cos x, y_p''(x) = 2A - D \cos x - E \sin x$$

Substituting into the problem, we have

$$4Ax^2 + 4Bx + (2A + 4C) + (-D + 4D) \cos x + (-E + 4E) \sin x = x^2 + \cos x$$

$$A = 1/4, B = 0, C = -1/8, D = 1/3, E = 0$$

$$\text{The complete solution is } y(x) = y_h(x) + y_p(x) = k_1 \cos 2x + k_2 \sin 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{3} \cos x$$

## b. Method of variation of parameters

The homogeneous solution of the equation (4.7) is

$$y_h(x) = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = \sum_{i=1}^n k_i y_i(x)$$

We replace the constants  $k_i, i = 1, 2, \dots, n$ , by  $u_i(x)$  to be determined, so that

$$y_p(x) = \sum_{i=1}^n u_i(x) y_i(x) \Rightarrow y'_p(x) = \sum_{i=1}^n u_i(x) y'_i(x) + \sum_{i=1}^n u'_i(x) y_i(x)$$

we choose

$$\sum_{i=1}^n u'_i(x) y_i(x) = 0 \quad (4.9)$$

then

$$y''_p(x) = \sum_{i=1}^n u_i(x) y''_i(x) + \sum_{i=1}^n u'_i(x) y'_i(x) \quad (4.10)$$

Choose

$$\sum_{i=1}^n u'_i(x) y'_i(x) = 0 \quad (4.11)$$

In a similar procedure, we have

$$\sum_{i=1}^n u'_i(x) y_i^{(n-2)}(x) = 0 \quad (4.12)$$

and  $y_p^{(n)}(x) = \sum_{i=1}^n u_i(x) y_i^{(n)}(x) + \sum_{i=1}^n u'_i(x) y_i^{(n-1)}(x)$

By substituting  $y_p$ , and its derivatives into (4.7), we have

$$\begin{aligned} \sum_{i=1}^n u_i(x) L y_i(x) + a_n \sum_{i=1}^n u'_i(x) y_i^{(n-1)}(x) &= r(x) \\ \sum_{i=1}^n u'_i(x) y_i^{(n-1)}(x) &= r(x) / a_n \end{aligned} \quad (4.12)$$

The conditions (4.9), (4.10), ..., (4.12) gives a system of  $n$  equations for the unknown functions  $u'_1, u'_2, \dots, u'_n$ :

$$\begin{aligned} \sum_{i=1}^n u'_i(x) y_i(x) &= 0 \\ \sum_{i=1}^n u'_i(x) y'_i(x) &= 0 \\ \dots & \\ \sum_{i=1}^n u'_i(x) y_i^{(n-2)}(x) &= 0 \\ \sum_{i=1}^n u'_i(x) y_i^{(n-1)}(x) &= r(x) / a_n \end{aligned}$$

where we can find  $u_1, u_2, \dots, u_n$

**Ex. 11**

$$y'' + y = \tan x.$$

Solution:  $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h(x) = k_1 \cos x + k_2 \sin x$

Let  $y_p(x) = u_1 \cos x + u_2 \sin x$ , then

$$u'_1 \cos x + u'_2 \sin x = 0$$

$$-u'_1 \sin x + u'_2 \cos x = \tan x$$

$$u'_1 = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{W(\cos x, \sin x)} = -\sin x \tan x$$

$$u_1 = \int -\sin x \tan x \, dx = \int \tan x \, d\cos x = \cos x \tan x - \int \cos x \sec^2 x \, dx = \sin x - \ln|\sec x + \tan x|$$

$$u'_2 = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{W(\cos x, \sin x)} = \sin x \Rightarrow u_2 = \int \sin x \, dx = -\cos x$$

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) = (k_1 + \sin x - \ln|\sec x + \tan x|) \cos x + (k_2 - \cos x) \sin x \\ &= (k_1 - \ln|\sec x + \tan x|) \cos x + k_2 \sin x \end{aligned}$$

**Ex. 12**

$$y''' + y' = \cot x.$$

Solution:  $\lambda^3 + \lambda = 0 \Rightarrow \lambda = 0, \pm i \Rightarrow y_h(x) = k_1 + k_2 \cos x + k_3 \sin x$

Let  $y_p(x) = u_1 + u_2 \cos x + u_3 \sin x$ , then

$$u'_1 + u'_2 \cos x + u'_3 \sin x = 0$$

$$-u'_2 \sin x + u'_3 \cos x = 0$$

$$-u'_2 \cos x - u'_3 \sin x = \cot x$$

$$u'_1 = \cot x, u'_2 = -\cot x \cos x, u'_3 = -\cos x$$

$$u_1 = \ln|\sin x|, u_2 = -\cos x - \ln|\csc x - \cot x|, u_3 = -\sin x$$

$$y(x) = y_h(x) + y_p(x) = k_1 + k_2 \cos x + k_3 \sin x - 1 - \cos x \ln|\csc x - \cot x| + \ln|\sin x|$$

### 3.Euler-Cauchy equations (Equidimensional linear differential equations)

Form:  $a_nx^n y^{(n)}(x) + a_{n-1}x^{n-1} y^{(n-1)}(x) + \dots + a_1xy'(x) + a_0y(x) = r(x)$  (4.13)

$$\text{Let } x = e^t \Rightarrow \frac{d}{dx} = \frac{dt}{dx} \frac{d}{dt} = \frac{1}{e^t} \frac{d}{dt}$$

$$x \frac{d}{dt} = \frac{d}{dt}$$

$$x^2 \frac{d^2}{dx^2} = x^2 \frac{d}{dx} \left( \frac{d}{dx} \right) = x^2 \frac{1}{e^t} \frac{d}{dt} \left( \frac{1}{e^t} \frac{d}{dt} \right) = e^t \left( \frac{1}{e^t} \frac{d^2}{dt^2} - \frac{1}{e^t} \frac{d}{dt} \right)$$

$$= \frac{d^2}{dt^2} - \frac{d}{dt} = \frac{d}{dt} \left( \frac{d}{dt} - 1 \right)$$

$$x^3 \frac{d^3}{dx^3} = e^{3t} \frac{1}{e^t} \frac{d}{dt} \left[ \frac{1}{e^t} \frac{d}{dt} \left( \frac{1}{e^t} \frac{d}{dt} \right) \right] = e^{2t} \frac{d}{dt} \left[ \frac{1}{e^t} \left( \frac{1}{e^t} \frac{d^2}{dt^2} - \frac{1}{e^t} \frac{d}{dt} \right) \right] = e^{2t} \frac{d}{dt} \left( \frac{1}{e^{2t}} \frac{d^2}{dt^2} - \frac{1}{e^{2t}} \frac{d}{dt} \right)$$

$$= e^{2t} \left( \frac{1}{e^{2t}} \frac{d^3}{dt^3} - 2 \frac{1}{e^{2t}} \frac{d^2}{dt^2} - \frac{1}{e^{2t}} \frac{d^2}{dt^2} + 2 \frac{1}{e^{2t}} \frac{d}{dt} \right) = \frac{d^3}{dt^3} - 3 \frac{d^2}{dt^2} + 2 \frac{d}{dt}$$

$$= \frac{d}{dt} \left( \frac{d}{dt} - 1 \right) \left( \frac{d}{dt} - 2 \right)$$

$$\dots$$

$$x^n \frac{d^n}{dt^n} = \frac{d}{dt} \left( \frac{d}{dt} - 1 \right) \left( \frac{d}{dt} - 2 \right) \left( \frac{d}{dt} - 3 \right) \dots \left( \frac{d}{dt} - n+1 \right)$$

The transformed equation thus becomes linear with constant coefficients, and  $y$  then can be determined in terms of  $t$ .

**Ex. 13**

$$x^2 y'' - 3xy' + 4y = x^2 \ln x.$$

Solution: Let  $x = e^t \Rightarrow t = \ln x$ , we get

$$\frac{d}{dt} \left( \frac{d}{dt} - 1 \right) y - 3 \frac{dy}{dt} + 4y = te^{2t} \Rightarrow \frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = te^{2t} \quad (4.14)$$

The characteristic equation is  $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2 \Rightarrow y_h = (k_1 + k_2 t)e^{2t}$

Assume particular solution is  $y_p = (At^3 + Bt^2)e^{2t}$

$$y_p' = [2At^3 + (3A + 2B)t^2 + 2Bt]e^{2t}$$

$$y_p'' = [4At^3 + (12A + 4B)t^2 + (6A + 8B)t + 2B]e^{2t}$$

$$(4.14) \Rightarrow [(4A - 8A + 4A)t^3 + (12A + 4B - 12A - 8B + 4B)t^2 + (6A + 8B - 8B)t + 2B]e^{2t} = te^{2t}$$

$$A = 1/6, B = 0$$

$$y = y_h + y_p = (k_1 + k_2 t + \frac{1}{6}t^3)e^{2t} = [k_1 + k_2 \ln x + \frac{1}{6}(\ln x)^3]x^2$$

**Ex. 14**

$$x^2y'' - 4xy' + 6y = -7x^4 \sin x.$$

Solution: Let  $x = e^t \Rightarrow t = \ln x$ , we get

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = -7e^{4t} \sin e^t \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3 \Rightarrow y_h = k_1 e^{2t} + k_2 e^{3t}$$

$$\text{Let } y_p = u_1 e^{2t} + u_2 e^{3t}$$

$$u'_1 e^{2t} + u'_2 e^{3t} = 0$$

$$2u'_1 e^{2t} + 3u'_2 e^{3t} = -7e^{4t} \sin e^t$$

$$u'_1 = \frac{\begin{vmatrix} 0 & e^{3t} \\ -7e^{4t} \sin e^t & 3e^{3t} \end{vmatrix}}{W(e^{2t}, e^{3t})} = 7e^{2t} \sin e^t \Rightarrow u_1 = -7e^t \cos e^t + 7 \sin e^t$$

$$u'_2 = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & -7e^{4t} \sin e^t \end{vmatrix}}{W(e^{2t}, e^{3t})} = -7e^t \sin e^t \Rightarrow u_2 = 7 \cos e^t$$

$$\begin{aligned} y &= y_h + y_p = k_1 e^{2t} + k_2 e^{3t} + (-7e^t \cos e^t + 7 \sin e^t) e^{2t} + 7e^{3t} \cos e^t \\ &= k_1 e^{2t} + k_2 e^{3t} + 7e^{2t} \sin e^t = k_1 x^2 + k_2 x^3 + 7x^2 \sin x \end{aligned}$$

4. Simultaneous linear differential equations

**Ex. 15**

$$\begin{cases} \frac{dx}{dt} + 2x + \frac{dy}{dt} + 6y = 2e^t \\ 2\frac{dx}{dt} + 3x + 3\frac{dy}{dt} + 8y = -1 \end{cases}$$

Solution: Let  $D = \frac{d}{dt} \Rightarrow \begin{cases} (D+2)x + (D+6)y = 2e^t \dots \dots \dots (1) \\ (2D+3)x + (3D+8)y = -1 \dots \dots \dots (2) \end{cases}$

$$(1) \times 2 - (2) \Rightarrow \begin{cases} (D+2)x + (D+6)y = 2e^t \dots \dots \dots (3) \\ x - (D-4)y = 4e^t + 1 \dots \dots \dots (4) \end{cases}$$

$$-(D+2) \times (4) + (3) \Rightarrow \begin{cases} (D^2 - D - 2)y = -10e^t - 2 \dots \dots \dots (5) \\ x - (D-4)y = 4e^t + 1 \dots \dots \dots (6) \end{cases}$$

The characteristic equation of (5) is  $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2 \Rightarrow y_h = k_1 e^{-t} + k_2 e^{2t}$

Assume  $y_p = Ae^t + B \Rightarrow y'_p = Ae^t \Rightarrow y''_p = Ae^t$

$$(5) \Rightarrow (A - A - 2A)e^t - 2B = -10e^t - 2 \Rightarrow A = 5, B = 1$$

$$y = y_h + y_p = k_1 e^{-t} + k_2 e^{2t} + 5e^t + 1$$

Substituting into (6), we get

$$x = (-k_1 - 4k_2)e^{-t} + (2k_2 - 4k_2)e^{2t} + (5 - 20)e^t - 4 + 4e^t + 1 = -5k_1 e^{-t} - 2k_2 e^{2t} - 11e^t - 3$$

$$\begin{cases} x \\ y \end{cases} = k_1 \begin{cases} -5 \\ 1 \end{cases} e^{-t} + k_2 \begin{cases} -2 \\ 1 \end{cases} e^{2t} + \begin{cases} -11 \\ 5 \end{cases} e^t + \begin{cases} -3 \\ 1 \end{cases}$$

**Ex. 16**

$$\begin{cases} \frac{dx}{dt} + x + \frac{dy}{dt} - y = \sec t \\ 4\frac{dx}{dt} + 5x + 5\frac{dy}{dt} - 4y = 5\sec t \end{cases}$$

The characteristic equation of (5) is  $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = k_1 \cos t + k_2 \sin t$

Assume  $y_p = u_1 \cos t + u_2 \sin t$

$$u'_1 \cos t + u'_2 \sin t = 0$$

$$-u'_1 \sin t + u'_2 \cos t = \sec t \tan t$$

$$u_1' = \begin{vmatrix} 0 & \sin t \\ \sec t \tan t & \cos t \end{vmatrix} = -\tan^2 t \Rightarrow u_1 = -\tan t + C$$

$$u_2' = \begin{vmatrix} \cos t & 0 \\ -\sin t & \sec t \tan t \end{vmatrix} = \tan t \Rightarrow u_2 = \ln |\sec t|$$

$$y = y_h + y_p = k_1 \cos t + k_2 \sin t - \sin t + t \cos t + \sin t \ln |\sec t|$$

Substituting into (6), we get

$$x = k_1 \sin t - k_2 \cos t + t \sin t - \cos t \ln |\sec t| - \sin t \tan t + \sec t$$

$$\begin{cases} x \\ y \end{cases} = k_1 \begin{cases} \sin t \\ \cos t \end{cases} + k_2 \begin{cases} -\cos t \\ \sin t \end{cases} + \begin{cases} t \sin t - \cos t \ln |\sec t| - \sin t \tan t + \sec t \\ -\sin t + t \cos t + \sin t \ln |\sec t| \end{cases}$$

## 5. Linear differential equations with variable coefficients

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = r(x)$$

## (1) Exact differential equations

Consider a simple case of equation (4.1):

$$\begin{aligned}
& p_3y''' + p_2y'' + p_1y' + p_0y = (p_3y'')' - p'_3y'' + p_2y'' + p_1y' + p_0y = (p_3y'')' + (p_2 - p'_3)y'' + p_1y' + p_0y \\
& = (p_3y'')' + [(p_2 - p'_3)y']' - (p'_2 - p_3'')y' + p_1y' + p_0y \\
& = (p_3y'')' + [(p_2 - p'_3)y']' + (p_1 - p'_2 + p_3'')y' + p_0y \\
& = (p_3y'')' + [(p_2 - p'_3)y']' + [(p_1 - p'_2 + p_3'')y]' - (p'_1 - p_2'' + p_3''')y + p_0y \\
& = (p_3y'')' + [(p_2 - p'_3)y']' + [(p_1 - p'_2 + p_3'')y]' + (p_0 - p'_1 + p_2'' - p_3''')y
\end{aligned}$$

If the coefficient of  $y$  is identically zero, the other three terms can be integrated, and we have a new equation of less order. In a similar way, equation (4.1) is said to be exact, if

$$\sum_{i=0}^n (-1)^i p_i^{(i)}(x) = 0$$

**Ex. 17**

$$(x-1)y'' + (x+1)y' + y = 2x.$$

Solution: Since  $p_0 - p'_1 + p_2'' = 0$ , the equation is exact.

$$[(x-1)y'] - y' + (x+1)y' + y = 2x$$

$$[(x-1)y'] + xy' + y = 2x$$

$$[(x-1)y'] + (xy)' = 2x \Rightarrow (x-1)y' + xy = x^2 + C_1$$

$$y' + \frac{x}{x-1}y = \frac{x^2 + C_1}{x-1}$$

This is a linear first order equation.

$$h = \int \frac{x}{x-1} dx = x + \ln(x-1)$$

$$y = e^{-h} \left[ \int \frac{x^2 + C_1}{x-1} e^h dx + C_2 \right] = \frac{e^{-x}}{x-1} \left[ \int e^x (x^2 + C_1) dx + C_2 \right]$$

$$= \frac{e^{-x}}{x-1} [x^2 - 2x + 2 + C_1] e^x + C_2$$

(2) Reduction of order

One of the important properties of linear differential equations is the fact that, if one homogeneous solution of an equation of order  $n$  is known, a new linear differential equation of order  $n-1$ , determining the remainder of the solution, can be obtained.

Suppose  $u(x)$  is the homogeneous solution, substituting  $y = v(x) u(x)$  into the original equation, we have a new equation of less order. We can try  $e^{mx}$ ,  $\cos mx$ ,  $\sin mx$ ,  $x^m$  for the homogeneous solution.

**Ex. 18**

$$xy'' + (x-1)y' - y = 0.$$

Solution: Since  $e^{-x}$  is a homogeneous solution, let  $y = ve^{-x}$

$$y' = v'e^{-x} - ve^{-x}, \quad y'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

$$x(v'' - 2v' + v)e^{-x} + (x-1)(v' - v)e^{-x} - ve^{-x} = 0$$

$$xv'' - (x+1)v' = 0$$

This is a first order equation in  $v'$

$$v' = C_1 e^{\int \frac{x+1}{x} dx} = C_1 x e^x$$

$$v = \int C_1 x e^x + C_2 = C_1 (x-1) e^x + C_2$$

$$y = C_1 (x-1) + C_2 e^{-x}$$

- [Exercises]
1.  $xy'' - (2x+1)y' + (x+1)y = x^2 - x - 1$
  2.  $(1+x^2)y'' + 2xy' - 2y = 3x$
  3.  $(x-1)y'' - xy' + y = 1$
  4.  $(1-x)y'' + xy' - y = (1-x)^2$

- [Answers]
1.  $y = C_1e^x + C_2x^2e^x + x$
  2.  $y = C_1x + C_2(1 + x\tan^{-1}x) + x\ln\sqrt{1+x^2}$
  3.  $y = C_1x + C_2e^x + 1$
  4.  $y = C_1x + C_2e^x + x^2 + 1$

## II. Nonlinear differential equations of higher order

1. Dependent variable absent:  $f(x, y', y'', \dots, y^{(n)}) = 0$

Let  $p = y'$ , the resulting equation is of order  $n-1$  in  $p$ .

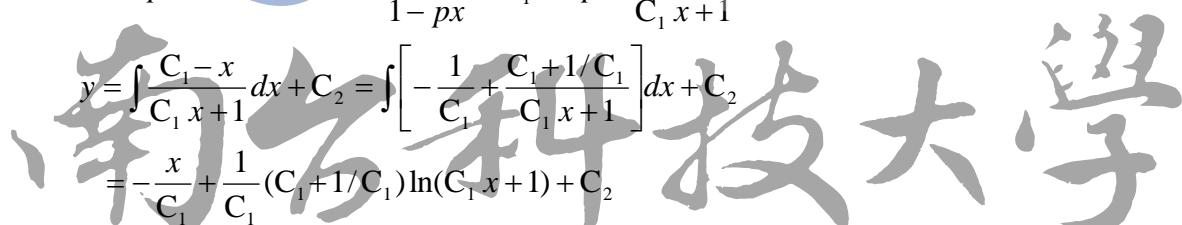
**Ex. 19**

$$(1+x^2)y'' + y'^2 + 1 = 0.$$

Solution: Let  $p = y' \Rightarrow (1+x^2)\frac{dp}{dx} + p^2 + 1 = 0 \Rightarrow \frac{dp}{p^2+1} + \frac{dx}{1+x^2} = 0$

$$\tan^{-1}p + \tan^{-1}x = C' \Rightarrow \frac{p+x}{1-px} = C_1 \Rightarrow p = \frac{C_1-x}{C_1x+1}$$

$$\begin{aligned} y &= \int \frac{C_1-x}{C_1x+1} dx + C_2 = \int \left[ -\frac{1}{C_1} + \frac{C_1+1/C_1}{C_1x+1} \right] dx + C_2 \\ &= -\frac{x}{C_1} + \frac{1}{C_1}(C_1+1/C_1) \ln(C_1x+1) + C_2 \end{aligned}$$



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- [Exercises]
1.  $\frac{d^2y}{dx^2} = x \left( \frac{dy}{dx} \right)^3$
  2.  $\frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 = 0$

- [Answers]
1.  $x = C_1 \sin(y - C_2)$
  2.  $y = \ln(x + C_1) + C_2$

2. Independent variable absent:  $f(y, y', y'', \dots, y^{(n)}) = 0$

Let  $p = y'$ ,  $y'' = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$ , ..., the equation can be reduced to an  $(n-1)$  order, where  $p$  is the new dependent variable and  $y$  is the new independent variable.

**Ex. 20**

$$yy'' + y'^2 - y' = 0.$$

Solution: Let  $p = y'$ ,  $y'' = p \frac{dp}{dy}$ , the equation reduces to

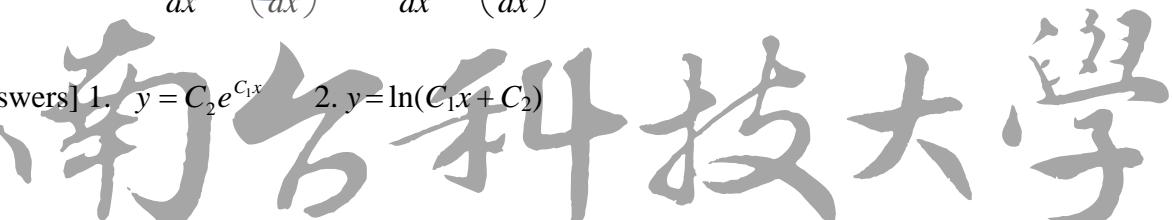
$$yp \frac{dp}{dy} + p^2 - p = 0 \Rightarrow \frac{dp}{p-1} + \frac{dy}{y} = 0 \Rightarrow \ln(p-1) + \ln y = C'$$

$$(p-1)y = C_1 \Rightarrow y \frac{dy}{dx} - y = C_1 \Rightarrow \frac{ydy}{C_1 + y} = dx \Rightarrow \left(1 - \frac{C_1}{C_1 + y}\right) dy = dx$$

$$y - C_1 \ln(y + C_1) = x + C_2$$

[Exercises] 1.  $y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$     2.  $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

[Answers] 1.  $y = C_2 e^{C_1 x}$     2.  $y = \ln(C_1 x + C_2)$



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3. Equations homogeneous in  $y$  and its derivatives

Let  $y = e^z \Rightarrow y' = e^z z'$ ,  $y'' = e^z(z'' + z'^2)$ , ...

$f(x, y, y', y'', \dots, y^{(n)}) = 0 \Rightarrow f(x, z', z'', \dots, z^{(n)}) = 0$  is the kind of dependent variable absent type.

Ex. 21

$$yy'' - y'^2 - 6xy^2 = 0.$$

Solution: Let  $y = e^z \Rightarrow y' = e^z z'$ ,  $y'' = e^z(z'' + z'^2)$

$$e^{2z}(z'' + z'^2) - z'^2 e^{2z} - 6x e^{2z} = 0 \Rightarrow z'' - 6x = 0 \Rightarrow z = x^3 + C_1 x + C_2$$

$$y = e^{x^3 + C_1 x + C_2}$$

[Exercises] 1.  $y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$     2.  $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

[Answers] 1.  $y = C_2 e^{C_1 x}$     2.  $y^2 = C_2 x + C_1$

#### 4. Isobaric equations

If a number  $m$  exists such that the result of replacing  $x$  by  $tx$ ,  $y$  by  $t^m y$ ,  $y'$  by  $t^{m-1} y'$ ,  $y''$  by  $t^{m-2} y''$ , ...,  $y^{(n)}$  by  $t^{m-n} y^{(n)}$ , is the original function multiplied by  $t^r$ , let  $y = ux^m$ ,  $x = e^t$ , we can simplify the original equation.

Ex. 22

$$x^2 yy'' - 3xyy' + x^2 y'^2 + 2y^2 + x^2 = 0.$$

Solution: weight:  $2 + m + (m - 2)$ ,  $1 + m + (m - 1)$ ,  $2 + 2(m - 1)$ ,  $2m$ ,  $2 \Rightarrow m = 1$

$$\begin{aligned} \text{Let } y = ux, x = e^t \Rightarrow y' &= \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \left(x \frac{du}{dt} + u \frac{dx}{dt}\right) e^{-t} = \frac{du}{dt} + u \\ y'' &= \frac{d}{dt} \left( \frac{du}{dt} + u \right) \frac{dx}{dt} = \left( \frac{d^2u}{dt^2} + \frac{du}{dt} \right) e^{-t} \end{aligned}$$

$$e^{2t} ue^t \left( \frac{d^2u}{dt^2} + \frac{du}{dt} \right) e^{-t} - 3e^t ue^t \left( \frac{du}{dt} + u \right) + e^{2t} \left( \frac{du}{dt} + u \right)^2 + 2u^2 e^{2t} + e^{2t} = 0$$

$$u \frac{d^2u}{dt^2} + \left( \frac{du}{dt} \right)^2 + 1 = 0 \Rightarrow \frac{d}{dt} \left( u \frac{du}{dt} \right) + 1 = 0 \Rightarrow u \frac{du}{dt} + t = C_1$$

$$udu = (C_1 - t) dt \Rightarrow \frac{u^2}{2} = -\frac{(C_1 - t)^2}{2} + C_2$$

$$\frac{1}{2} \left( \frac{y}{x} \right)^2 = -\frac{(C_1 - \ln x)^2}{2} + C_2$$

[Exercises] 1. Show that the substitution  $y = u' / [Q(x)u]$ , where  $u' = du/dx$ , reduces the nonlinear first-order equation

$$y' + P(x)y + Q(x)y^2 = R(x)$$

to the second-order equation

$$\frac{d^2u}{dx^2} + \left( P - \frac{Q'}{Q} \right) \frac{du}{dx} - RQu = 0$$

and also that the substitution  $y = y_1 + 1/v$ , where  $y_1(x)$  is any known solution of the given equation, leads to the linear first-order equation

$$v' - (P + 2Qy_1)v = Q.$$

(The nonlinear form is known as Riccati's equation.)

2. Use the procedures suggested above to obtain the general solution of the equation  $x^2y' + xy + x^2y^2 = 1$  in the form  $(x^2 - k)/(x^2 + k)$ , where  $k$  is an arbitrary constant.

