Chapter 2 Properties of Laplace Transform

I. Properties of Laplace Transform

Property	Original Function	Transformed Function
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$
Shifting	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
	$e^{at}f(t)$	$F(s-a)$
Scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{(n-1)}(0)$
	$(-t)^nf(t)$	$\frac{d^nF(s)}{ds^n}$
Integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)ds$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
Periodic Function	$f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$

1. Linearity

$$\mathrm{L}\,[af(t)+bg(t)]=\int_0^\infty [af(t)+bg(t)]e^{-st}dt=a\int_0^\infty f(t)e^{-st}dt+b\int_0^\infty g(t)e^{-st}dt=aF(s)+bG(s)$$

Ex. 1

Find the Laplace transform of $\cos^2 t$.

Solution: $\mathrm{L}\,[\cos^2 t]=\mathrm{L}\,[\frac{1+\cos 2t}{2}]=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right)=\frac{s^2+2}{s(s^2+4)}$

2. Shifting

(a) $\mathcal{L}\,[f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$

Let $\tau = t-a$, then

$$\mathcal{L}\,[f(t-a)u(t-a)] = \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sa}F(s)$$

(b) $F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty [e^{at}f(t)]e^{-st}dt = \mathcal{L}\,[e^{at}f(t)]$

Ex. 2

What is the Laplace transform of the function: $f(t) = \begin{cases} 0, & t<4 \\ 2t^3, & t \ge 4 \end{cases}$.

Solution: $f(t) = 2t^3u(t-4)$

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,\{2[(t-4)^3 + 12(t-4)^2 + 48(t-4) + 64]\,u(t-4)\}$$

$$= 2e^{-4s}\left(\frac{3!}{s^4} + 12\times\frac{2!}{s^3} + 48\times\frac{1}{s^2} + \frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4} + \frac{12}{s^3} + \frac{24}{s^2} + \frac{32}{s}\right)$$

3. Scaling

$$\mathcal{L}\,[f(at)] = \int_0^\infty f(at)e^{-st}dt$$

Let $\tau = at$, then

$$\mathcal{L}\,[f(at)] = \int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}d\frac{\tau}{a} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F(\frac{s}{a})$$

Ex. 3

Find the Laplace transform of cos2*t*.

Solution: $\because \mathcal{L}\,[\cos t] = \dfrac{s}{s^2+1}$

$$\therefore \mathcal{L}\,[\cos 2t] = \frac{1}{2}\,\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

4. Derivative

(*a*) Derivative of original function

$$\mathcal{L}\,[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt \qquad (2.1)$$

(1) If $f(t)$ is continuous, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = -f(0) + sF(s) = sF(s) - f(0)$$

(2) If $f(t)$ is not continuous at $t = a$, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = f(t)e^{-st}\Big|_0^{a^-} + f(t)e^{-st}\Big|_{a^+}^\infty + sF(s) = [f(a^-)\,e^{-sa} - f(0)] + [0 - f(a^+)\,e^{-sa}] + sF(s)$$

$$= sF(s) - f(0) - e^{-sa}\,[f(a^+) - f(a^-)]$$

(3) Similarly, if $f(t)$ is not continuous at $t = a_1, a_2, \cdots,\cdots, a_n$, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = sF(s) - f(0) - \sum_{i=1}^{n} e^{-sa_i}[\,f(a_i^+) - f(a_i^-)]$$

[Deduction] If $f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L}\,[f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

(*b*) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}[\,f(t)e^{-st}]dt = \int_0^\infty (-t)\,f(t)e^{-st}dt = \mathcal{L}\;[(-t)\,f(t)]$$

[Deduction] $\dfrac{d^nF(s)}{ds^n} = \mathcal{L}\;[(-t)^n f(t)]$

Ex. 4

Find the Laplace transform of te^t.

Solution : $\mathcal{L}\;(e^t) = \dfrac{1}{s-1} \Rightarrow \mathcal{L}\;(te^t) = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) = \dfrac{1}{(s-1)^2}$

Ex. 5

$f(t)=\begin{cases} t^2, & 0\le t\le 1 \\ 0, & t>1 \end{cases}$, find $\mathcal{L}\,[f'(t)]$.

Solution : $f(t)=t^2[u(t)-u(t-1)]$

$$\mathcal{L}\,[f(t)]=\mathcal{L}\,[t^2u(t)]-\mathcal{L}\,[t^2u(t-1)]=\frac{2!}{s^3}-\mathcal{L}\,\{[(t-1)+1]^2u(t-1)\}$$

$$=\frac{2}{s^3}-\mathcal{L}\,\{[(t-1)^2+2(t-1)+1]u(t-1)\}$$

$$=\frac{2}{s^3}-e^{-s}(\frac{2}{s^3}+2\frac{1}{s^2}+\frac{1}{s})$$

$$\mathcal{L}\,[f'(t)]=sF(s)-f(0)-e^{-s}[f(1^+)-f(1^-)]$$

$$=[\frac{2}{s^2}-e^{-s}(\frac{2}{s^2}+\frac{2}{s}+1)]-0-e^{-s}(0-1)=\frac{2}{s^2}-e^{-s}(\frac{2}{s^2}+\frac{2}{s})$$

5. Integration

(a) Integral of original function

$$\mathcal{L}\,[\int_0^t f(\tau)d\tau]=\int_0^\infty\int_0^t f(\tau)d\tau e^{-st}dt$$

$$=\frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty-\int_0^\infty f(t)e^{-st}dt\right]=\frac{1}{s}F(s)$$

$$\Rightarrow\mathcal{L}\,[\int_0^t\int_0^t\ldots\int_0^t f(t)dtdt\cdots dt]=\frac{1}{s^n}F(s)$$

(b) Integration of Laplace transform

$$\int_s^\infty F(s)ds=\int_s^\infty\int_0^\infty f(t)e^{-st}dtds=\int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$=\int_0^\infty f(t)\frac{e^{-st}}{-t}\Big|_s^\infty dt=\int_0^\infty\frac{f(t)}{t}e^{-st}dt=\mathcal{L}\,[\frac{f(t)}{t}]$$

$$\Rightarrow\int_s^\infty\int_s^\infty\cdots\int_s^\infty F(s)dsds\cdots ds=\mathcal{L}\,[\frac{1}{t^n}f(t)]$$

Ex. 6

Find (a) $\mathcal{L}\left[\dfrac{1-e^{-t}}{t}\right]$ (b) $\mathcal{L}\left[\dfrac{1-e^{-t}}{t^2}\right]$.

Solution : (a) $\mathcal{L}[1-e^{-t}] = \dfrac{1}{s} - \dfrac{1}{s+1}$

$$\mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right)ds = \ln s - \ln(s+1)\Big|_s^{\infty} = \ln\frac{s}{s+1}\Big|_s^{\infty}$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

(b) $$\mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right] = \int_s^{\infty}\ln\frac{s+1}{s}\,ds = s\ln\frac{s+1}{s}\Big|_s^{\infty} - \int_s^{\infty} s\left(\frac{1}{s+1}-\frac{1}{s}\right)ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^{\infty} + \int_s^{\infty}\frac{1}{s+1}\,ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^{\infty}$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^{\infty} = s\ln s - (s+1)\ln(s+1)$$

Ex. 7

Find (a) $\displaystyle\int_0^{\infty}\frac{\sin kt\, e^{-st}}{t}\,dt$ (b) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$.

Solution : (a) $\displaystyle\int_0^{\infty}\frac{\sin kte^{-st}}{t}\,dt = \mathcal{L}\left[\frac{\sin kt}{t}\right]$

$$\because \mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\left[\frac{\sin kt}{t}\right] = \int_s^{\infty}\frac{k}{s^2+k^2}\,ds = \frac{1}{k}\int_s^{\infty}\frac{1}{(\frac{s}{k})^2+1}\,ds$$

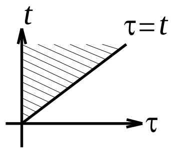
$$= \tan^{-1}\frac{s}{k}\Big|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

(b) $$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = 2\int_0^{\infty}\frac{\sin x}{x}\,dx$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}}\int_0^{\infty}\frac{\sin kte^{-st}}{t}\,dt$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{k}\right) = \pi$$

6. Convolution theorem



$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dt d\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}dt d\tau$$

Let $u = t-\tau$, $du = dt$, then

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty f(\tau)\int_0^\infty g(u)e^{-s(u+\tau)}du d\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau \int_0^\infty g(u)e^{-su}du = F(s)G(s)$$

Ex. 8

Find the Laplace transform of $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$.

Solution : $\because \mathcal{L}[e^t] = \dfrac{1}{s-1}$, $\mathcal{L}[\sin 2t] = \dfrac{2}{s^2+4}$

$$\therefore \mathcal{L}\left[\int_0^t e^{t-\tau}\sin 2t\, d\tau\right] = \mathcal{L}[e^t * \sin 2t] = \mathcal{L}[e^t]\cdot\mathcal{L}[\sin 2t]$$

$$= \frac{1}{s-1}\cdot\frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: $f(t+T) = f(t)$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

and $\displaystyle\int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore \mathcal{L}[f(t)] = (1+e^{-sT}+e^{-2sT}+\cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

Ex. 9

Find the Laplace transform of $f(t)=\frac{k}{p}t,\ 0<t<p,\ f(t+p)=f(t)$.

Solution : $L\ [f(t)]=\frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$

$$=\frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p-\int_0^p e^{-st}dt)]$$

$$=\frac{-k}{ps(1-e^{-ps})}(te^{-st}+\frac{1}{s}e^{-st})\Big|_0^p$$

$$=\frac{-k}{ps(1-e^{-ps})}(pe^{-sp}+\frac{e^{-sp}}{s}-\frac{1}{s})$$

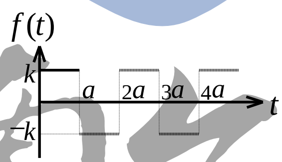
[Exercises] Find the Laplace transform of the problems:1–4, and 7

1. $e^{-\alpha t}(A\cos\beta t+B\sin\beta t)$ 2. $t^2\cos t$ 3. $u(t-\pi)\cos t$

4. $\int_t^\infty \frac{\cos x}{x}dx$ (cosine integral function) 5. Find the value of the integral $\int_0^\infty te^{-2t}\cos t dt$

6. Find the value of the integral $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$

7.



[Ans.] 1. $\frac{A(s+\alpha)+B\beta}{(s+\alpha)^2+\beta^2}$ 2. $\frac{2s(s^2-3)}{(s^2+1)^3}$ 3. $\frac{-se^{-\pi s}}{s^2+1}$

4. $\frac{\ln(s^2+1)}{2s}$ 5. $\frac{3}{25}$ 6. $\ln 3$ 7. $\frac{k}{s}\tanh\frac{as}{2}$