Chapter 2 Differential Calculus of Vector

I. Derivative of a vector

Definition: A vector function $\mathbf{v}(t)$ is said to be differentiable at a point t if the limit

$$\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$

exists. The vector $\mathbf{v}'(t)$ is called the derivative of $\mathbf{v}(t)$. In terms of components with respect to a given Cartesian coordinate system, the derivative $\mathbf{v}'(t)$ is obtained by differentiating each component separately

$$\mathbf{v}'(t) = v'_1(\mathrm{t})\mathbf{i} + v'_2(\mathrm{t})\mathbf{j} + v'_2(\mathrm{t})\mathbf{k}$$

It follows

$$(\mathbf{u}+\mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u}\cdot\mathbf{v})' = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'$$

$$(\mathbf{u}\times\mathbf{v})' = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$$

Ex. 1.

Let $\mathbf{v}(t)$ be a vector function, whose length is constant. Then $\mathbf{v}\cdot\mathbf{v} = v^2 \Rightarrow (\mathbf{v}\cdot\mathbf{v})' = 0$, we get $2\mathbf{v}\cdot\mathbf{v}' = 0$. This yields the important result: the derivative of a vector function $\mathbf{v}(t)$ of constant length is either the zero vector or is perpendicular to $\mathbf{v}(t)$.

II. Geometry of a space curve

1. Arc length

A Cartesian coordinate system being given, we may represent a curve C by a vector function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}$$

If s represents arc length along the curve

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)\frac{ds}{dt}$$

The expression in parentheses is clearly a unit vector tangent to the curve at point P, we denote this unit vector by $\mathbf{T}$, and

$$\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{v}$$

$$|\mathbf{v}| = |\mathbf{T}|\frac{ds}{dt} = \frac{ds}{dt}$$

The arc length is

$$s = \int ds = \int |\mathbf{v}|\,dt = \int \sqrt{\mathbf{v}\cdot\mathbf{v}}\,dt \qquad (2.1)$$

Ex. 2.

Find the tangent to the ellipse $\frac{1}{4}x^2 + y^2 = 1$ at $P(\sqrt{2}, \frac{1}{\sqrt{2}})$.

Solution: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, and P corresponds to $t = \pi/4$

$$\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

$$\mathbf{r}'(\pi/4) = -\sqrt{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

Ex. 3.

Find the arc length of a circular helix: $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k}$, $(c \neq 0)$ $\quad 0 \le t \le b$.

Solution: $\mathbf{v}(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + c\,\mathbf{k}$

$$\mathbf{v}\cdot\mathbf{v} = a^2 + c^2$$

$$s = \int_0^b \sqrt{a^2 + c^2}\,dt = b\sqrt{a^2 + c^2}$$

Ex. 4.

Find the length of the space curve: $y = \sin 2\pi x$, $z = \cos 2\pi x$, from (0, 0, 1) to (1, 0, 1).

Solution: $\mathbf{r} = x\,\mathbf{i} + \sin 2\pi x\,\mathbf{j} + \cos 2\pi x\,\mathbf{k} \Rightarrow d\mathbf{r}/dx = \mathbf{i} + 2\pi\cos 2\pi x\,\mathbf{j} - 2\pi\sin 2\pi x\,\mathbf{k}$

$$s = \int_0^1 \sqrt{1^2 + (2\pi\cos 2\pi x)^2 + (2\pi\sin 2\pi x)^2}\,dx = \int_0^1 \sqrt{1 + 4\pi^2}\,dx = \sqrt{1 + 4\pi^2}$$

Ex. 5.

Find the length of the curve: $r = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, $a > 0$.

Solution: $\mathbf{r} = r\mathbf{e}_r \Rightarrow \frac{d\mathbf{r}}{d\theta} = \frac{dr}{d\theta}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{d\theta} = a\sin\theta\,\mathbf{e}_r + r\,\mathbf{e}_\theta = a\sin\theta\,\mathbf{e}_r + a(1 - \cos\theta)\,\mathbf{e}_\theta$

$$s = \int_0^{2\pi} \sqrt{(a\sin\theta)^2 + [a(1 - \cos\theta)]^2}\,d\theta = \int_0^{2\pi} \sqrt{2a^2 - 2a^2\cos\theta}\,d\theta = a\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\,d\theta$$

$$= a\int_0^{2\pi} 2\sin\frac{\theta}{2}\,d\theta = 4a(-\cos\frac{\theta}{2})\Big|_0^{2\pi} = 8a$$

2. Frenet formula

We have a unit tangent vector $\mathbf{T} = \dfrac{d\mathbf{r}}{ds}$, and define

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \kappa \ge 0 \tag{2.2}$$

where κ is the magnitude of $\dfrac{d\mathbf{T}}{ds}$ and is called the curvature, and $\mathbf{N}$ is a unit vector normal to $\mathbf{T}$ and is called the normal or principal normal to the curve. The reciprocal of the curvature, $\rho = \dfrac{1}{\kappa}$, is called the radius of curvature. In addition to the two perpendicular vector $\mathbf{T}$ and $\mathbf{N}$, a third unit vector known as the binormal vector, is definede by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

then

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \kappa\mathbf{N}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$$

Since $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$, and from the right hand side of the above equation, $\dfrac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$, $\dfrac{d\mathbf{B}}{ds}$ can be written in the form

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \tag{2.3}$$

where τ is the torsion or twisting of the curve.

$$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} = -\tau\mathbf{N}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B} \tag{2.4}$$

The three formulas, equations (2.2), (2.3), and (2.4) are called Frenet formulas.

In general, the position vector $\mathbf{r}$ is a function of t, to find κ and τ, we differentiate $\mathbf{r}$ with respect to t:

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\mathbf{T}$$

$$\ddot{\mathbf{r}} = \dot{v}\mathbf{T} + v\frac{d\mathbf{T}}{dt} = \dot{v}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \dot{v}\mathbf{T} + \kappa v^2\mathbf{N}$$

$$\dddot{\mathbf{r}} = \ddot{v}\mathbf{T} + \dot{v}\frac{d\mathbf{T}}{dt} + \frac{d}{dt}(\kappa v^2)\mathbf{N} + \kappa v^2\frac{d\mathbf{N}}{dt} = \ddot{v}\mathbf{T} + \dot{v}\kappa v\mathbf{N} + \frac{d}{dt}(\kappa v^2)\mathbf{N} + \kappa v^3(-\kappa\mathbf{T} + \tau\mathbf{B})$$

$$= (\ddot{v} - \kappa^2 v^3)\mathbf{T} + [\kappa v\dot{v} + \frac{d}{dt}(\kappa v^2)]\mathbf{N} + \kappa v^3\tau\mathbf{B}$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \kappa v^3\mathbf{B} \Rightarrow \kappa = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \tag{2.5}$$

$$(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}} = \kappa^2 v^6\tau \Rightarrow \tau = \frac{(\dot{\mathbf{r}}\;\ddot{\mathbf{r}}\;\dddot{\mathbf{r}})}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}$$

Ex. 6.

Given a space curve $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k}$, find (1)curvature (2)torsion (3)unit tangent, normal, and binormal vectors.

Solution : $\mathbf{v} = \dot{\mathbf{r}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k} = v\mathbf{T} \Rightarrow v = \sqrt{a^2+c^2},\ \dot{v} = 0$

$$\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{-a\sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = -a\cos t\mathbf{i} - a\sin t\mathbf{j} = \dot{v}\mathbf{T} + \kappa v^2\mathbf{N} = \kappa(a^2+c^2)\mathbf{N},\ \dddot{\mathbf{r}} = a\sin t\mathbf{i} - a\cos t\mathbf{j}$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \kappa v^3\mathbf{B} \Rightarrow ac\sin t\mathbf{i} - ac\cos t\mathbf{j} + a^2\mathbf{k} = \kappa v^3\mathbf{B}$$

$$\kappa v^3 = \sqrt{a^2c^2 + a^4} = a\sqrt{a^2+c^2} \Rightarrow \kappa = \frac{a}{a^2+c^2}$$

Substituting into $\mathbf{a}$, we have $\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$$\tau = \frac{(\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}})}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{a^2c}{a^2(a^2+c^2)} = \frac{c}{a^2+c^2}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2+c^2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{a^2+c^2}}(c\sin t\mathbf{i} - c\cos t\mathbf{j} + a\mathbf{k})$$

Ex. 7.

Show that for a curve $y = y(x)$ in the x-y plane, $\kappa(x) = \dfrac{|y''|}{(1+y'^2)^{3/2}}\ (y' = \dfrac{dy}{dx})$

Solution: $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$

$$\mathbf{v} = \mathbf{i} + y'\,\mathbf{j} = v\,\mathbf{T},\ \Rightarrow v = \sqrt{1+y'^2}$$

$$\mathbf{a} = y''\mathbf{j} = \dot{v}\,\mathbf{T} + \kappa v^2\mathbf{N}$$

$$\mathbf{v} \times \mathbf{a} = \kappa v^3\mathbf{B} \Rightarrow y''\mathbf{k} = \kappa v^3\mathbf{B} \Rightarrow \kappa = \frac{|y''|}{v^3} = \frac{|y''|}{(1+y'^2)^{3/2}}$$

[Exercise] The position vector of a particle P is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2t^3/3\,\mathbf{k}$. Find (1) the velocity, acceleration, and speed of P, (2) the tangential and centripetal components of acceleration, (3) the curvature of the curve C traversed by P, and (4) the minimum radius of curvature of C.

[Answer] (1) $\mathbf{v} = \mathbf{i} + 2t\,\mathbf{j} + 2t^2\,\mathbf{k}$, $\mathbf{a} = 2\mathbf{j} + 4t\,\mathbf{k}$, $v = 1 + 2t^2$ (2) $a_t = 4t$, $a_n = 2$ (3) $\kappa = 2/(1+2t^2)^2$ (4)

$\rho_{min} = 1/2$

III. Gradient

Definition: For a given scalar function $f(x, y, z)$ the gradient of f is the vector function

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$, read *del*, is a differential operator. Then

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \nabla f \cdot \frac{d\mathbf{r}}{ds} = \nabla f \cdot \mathbf{T}$$

where $\mathbf{T}$ is the unit tangent vector to the curve described by $\mathbf{r}(s)$. Since $\nabla f \cdot \mathbf{T} = |\nabla f| \cos\theta$, where θ is the angle between ∇f and $\mathbf{T}$, then $\frac{df}{ds} = |\nabla f| \cos\theta$; so the component of ∇f in the direction of $\mathbf{T}$ is the directional derivative of f in that direction. It readily follows from this result that ∇f is in the direction in which $\frac{df}{ds}$ has its maximum values; this maximum value is equal to $|\nabla f|$.

Consider a surface S in space given by $f(x, y, z) = \text{constant}$. As long as we move along this surface on any curve C, f has constant value and consequently $\frac{df}{ds} = 0$. Thus

$$\frac{df}{ds} = \nabla f \cdot \frac{d\mathbf{r}}{ds} = 0 \Rightarrow \nabla f \perp \frac{d\mathbf{r}}{ds} \Rightarrow \nabla f \perp \mathbf{T}$$

where $\mathbf{r}(s)$ is the position vector describing the curve C. Since ∇f is completely determined by f and C is any curve on the surface, ∇f must be normal to the surface.

Ex. 8.

Find a unit normal vector $\mathbf{n}$ of the cone of revolution $z^2 = 4(x^2 + y^2)$ at point P(1, 0, 2).

Solution: The cone is the level surface $f = 0$ of $f(x, y, z) = 4(x^2 + y^2) - z^2$. Thus

$$\nabla f = 8x\,\mathbf{i} + 8y\,\mathbf{j} - 2z\,\mathbf{k}$$

and at P

$$\nabla f|_{(1,0,2)} = 8\mathbf{i} - 4\mathbf{k}$$

$$\mathbf{n} = \left.\frac{\nabla f}{|\nabla f|}\right|_{(1,0,2)} = \frac{8\mathbf{i} - 4\mathbf{k}}{\sqrt{8^2 + (-4)^2}} = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{k}$$

Ex. 9.

Find the directional derivative of $f(x, y, z) = 4x^2y + 3y^2 + 2xz^2$ at the point $(1, -1, 0)$ in the direction of the vector $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j}$.

Solution: $\nabla f = (8xy + 2z^2)\,\mathbf{i} + (4x^2 + 6y)\,\mathbf{j} + 4xz\,\mathbf{k}$

$$\nabla f|_{(1,\,-1,\,0)} = -8\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{e}_b = \frac{\mathbf{b}}{|\mathbf{b}|} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\nabla f \cdot \mathbf{e}_b = 16/5$$

[Exercise] The temperature at any point in space is given by $T = xy + yz + zx$.

(1) Find the direction cosines of the direction in which the temperature changes most rapidly with distance from the point (1, 1, 1), and determine the maximum rate of change.

(2) Find the derivative of T in the direction of the vector $3\mathbf{i} - 4\mathbf{k}$ at the point (1, 1, 1).

[Answer] (1)$\pm(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$; $2\sqrt{3}$ (2)$-2/5$

IV. Divergence of a vector field

Definition: Let $\mathbf{v}(x, y, z)$ be a differentiable vector function and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then the function

$$\text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\mathbf{v}$.

For a small rectangular solid with edges Δx, Δy, and Δz. In the y direction the outflow of $\mathbf{v}$ through the right-hand face is $v_2(x, y + \Delta y, z)\Delta x\Delta z$, through the left-hand face is $-v_2(x, y, z)\,\Delta x\Delta z$, the total outflow in the y direction is the sum

$$[v_2(x, y + \Delta y, z) - v_2(x, y, z)]\Delta x\Delta z$$

The difference in these values of v_2 is

$$\frac{\partial v_2}{\partial y}\Delta y$$

Thus the net outflow from the two faces is

$$\frac{\partial v_2}{\partial y}\Delta x\Delta y\Delta z$$

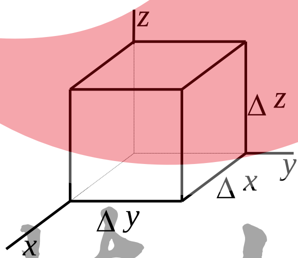
Similarly, the two faces in the x and z direction contribute respectively

$$\frac{\partial v_1}{\partial x}\Delta x\Delta y\Delta z \quad \text{and} \quad \frac{\partial v_3}{\partial z}\Delta x\Delta y\Delta z$$

After we divide by the volume $\Delta x\Delta y\Delta z$, we have

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

as the net outflow per unit volume.



Ex. 10.

If $\mathbf{F} = x\,\mathbf{i} + y^2 z\,\mathbf{j} + xz^3\,\mathbf{k}$, find $\nabla \cdot \mathbf{F}$.

Solution: $\nabla \cdot \mathbf{F} = 1 + 2yz + 3xz^2$

Ex. 11.

Find $\nabla \cdot \mathbf{r}$, given $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution: $\nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3$

V. Curl of a vector field

Definition: Let $\mathbf{v}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function. Then the function

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

is called the curl of $\mathbf{v}$ or the rotation of $\mathbf{v}$.

Ex. 12.

Let $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$ represent the angular velocity of a rigid body, the velocity field is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ $= (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. The curl of $\mathbf{v}$ is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} = 2\boldsymbol{\omega}$$

Hence, the curl of the velocity field in the case of the rotation of a rigid body has the direction of the axis of rotation, and its magnitude equals twice the angular speed $\boldsymbol{\omega}$ of the rotation.

VI. Vector identities

$$\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$$
$$\nabla\cdot(\phi\mathbf{F}) = \nabla\phi\cdot\mathbf{F} + \phi\nabla\cdot\mathbf{F}$$
$$\nabla\times(\phi\mathbf{F}) = \nabla\phi\times\mathbf{F} + \phi\nabla\times\mathbf{F}$$
$$\nabla f(u) = \frac{df}{du}\nabla u$$
$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$$
$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + (\nabla\cdot\mathbf{G})\mathbf{F} - (\nabla\cdot\mathbf{F})\mathbf{G}$$
$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$$
$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$
$$\nabla\cdot\mathbf{r} = 3$$
$$\nabla\times\mathbf{r} = 0$$
$$\nabla\times(\nabla\phi) = 0$$
$$\nabla\cdot(\nabla\times\mathbf{F}) = 0$$
$$\nabla\cdot(\nabla\phi_1\times\nabla\phi_2) = 0$$

Tensor notations:

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i$$
$$\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk}\, a_j\, b_k$$
$$\nabla\phi = \phi_{,i}$$
$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = \phi_{,ii}$$
$$\nabla\cdot\mathbf{F} = F_{i,i}$$
$$\nabla\times\mathbf{F} = \varepsilon_{ijk}\, F_{k,j}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, & (i,\ j,\ k) = (1,2,3),\ (2,3,1),\ (3,1,2) \\ -1, & (i,\ j,\ k) = (1,3,2),\ (2,1,3),\ (3,2,1) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varepsilon_{ijk}\varepsilon_{ips} = \delta_{jp}\,\delta_{ks} - \delta_{js}\,\delta_{kp}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Ex. 13.

Prove $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}.$

Solution: $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})_j\mathrm{C}_k=\varepsilon_{ijk}\,\varepsilon_{jlm}\,\mathrm{A}_l\mathrm{B}_m\mathrm{C}_k$

$$=\varepsilon_{jki}\,\varepsilon_{jlm}\,\mathrm{A}_l\mathrm{B}_m\mathrm{C}_k=(\delta_{kl}\,\delta_{im}-\delta_{km}\,\delta_{il})\mathrm{A}_l\mathrm{B}_m\mathrm{C}_k$$

$$=\mathrm{A}_k\mathrm{B}_i\mathrm{C}_k-\mathrm{A}_i\mathrm{B}_k\mathrm{C}_k$$

$$=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

Ex. 14.

Prove $\nabla\times(\mathbf{F}\times\mathbf{G})=(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}+(\nabla\cdot\mathbf{G})\mathbf{F}-(\nabla\cdot\mathbf{F})\mathbf{G}.$

Solution: $\nabla\times(\mathbf{F}\times\mathbf{G})=\varepsilon_{ijk}(\mathbf{F}\times\mathbf{G})_{k,j}=\varepsilon_{ijk}\,\varepsilon_{klm}(\mathrm{F}_l\mathrm{G}_m)_{,j}$

$$=\varepsilon_{kij}\,\varepsilon_{klm}(\mathrm{F}_l\mathrm{G}_m)_{,j}=(\delta_{il}\,\delta_{jm}-\delta_{im}\,\delta_{jl})(\mathrm{F}_l\mathrm{G}_m)_{,j}$$

$$=(\mathrm{F}_i\mathrm{G}_j)_{,j}-(\mathrm{F}_j\mathrm{G}_i)_{,j}$$

$$=\mathrm{F}_{i,j}\mathrm{G}_j+\mathrm{F}_i\mathrm{G}_{j,j}-\mathrm{F}_{j,j}\mathrm{G}_i-\mathrm{F}_j\mathrm{G}_{i,j}$$

$$=(\mathbf{G}\cdot\nabla)\mathbf{F}+(\nabla\cdot\mathbf{G})\mathbf{F}-(\nabla\cdot\mathbf{F})\mathbf{G}-(\mathbf{F}\cdot\nabla)\mathbf{G}$$

Ex. 15.

Prove $\nabla\times(\nabla\phi)=0.$

Solution: $\nabla\times(\nabla\phi)=\varepsilon_{ijk}(\nabla\phi)_{k,j}=\varepsilon_{ijk}\,\phi_{,kj}$

Since $\phi_{,kj}=\phi_{,jk}$ and $\varepsilon_{ijk}=-\varepsilon_{ikj}$

$\therefore\quad \varepsilon_{ijk}\phi_{,kj}=0$

Ex. 16.

Prove $(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{1}{2}\nabla|\mathbf{v}|^2-\mathbf{v}\times(\nabla\times\mathbf{v}).$

Solution: $\mathbf{v}\times(\nabla\times\mathbf{v})=\varepsilon_{ijk}\,v_j(\nabla\times\mathbf{v})_k=\varepsilon_{ijk}\,v_j(\varepsilon_{klm}\,v_{m,l})$

$$=\varepsilon_{kij}\,\varepsilon_{klm}\,v_jv_{m,l}=(\delta_{il}\,\delta_{jm}-\delta_{im}\,\delta_{jl})v_jv_{m,l}$$

$$=v_jv_{j,i}-v_jv_{i,j}=\frac{1}{2}(v_jv_j)_{,i}-v_jv_{i,j}$$

$$=\frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v})-(\mathbf{v}\cdot\nabla)\mathbf{v}$$

$$=\frac{1}{2}\nabla|\mathbf{v}|^2-(\mathbf{v}\cdot\nabla)\mathbf{v}$$

$$\therefore\ (\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{1}{2}\nabla|\mathbf{v}|^2-\mathbf{v}\times(\nabla\times\mathbf{v})$$

[Exercise] Show that $\nabla(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\nabla\mathbf{v} + 2\mathbf{v}\times(\nabla\times\mathbf{v})$.