Vector Analysis

Chapter 1 Algebra of Vectors

I. Elementary properties of vectors

Consider a Cartesian coordinate system, in which we define unit vectors, **i**, **j,** and **k** having the directions of positive x, y, and z axes, respectively. Any vector **v** whose projections on these axes are v_x, v_y, and v_z, can be written as

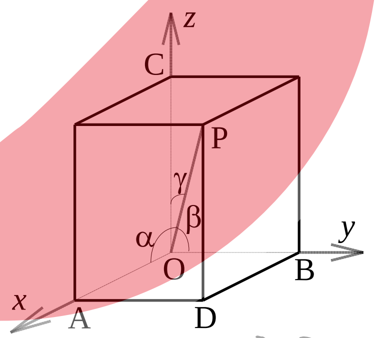
$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \quad \text{or} \quad \mathbf{v} = [v_x, v_y, v_z]$$

The number v_x, v_y, and v_z are called the scalar components of **v** in the x, y, and z directions. If, when the initial point of **v** coincides with the origin, the angle measured to **v** from the positive x, y, and z axes are denoted by α, β, and γ, respectively, there follows

$$v_x = v\cos\alpha,\ v_y = v\cos\beta,\ v_z = v\cos\gamma,$$

where $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the magnitude or length of **v**, the number cosα, cosβ, and cosγ are called the direction cosines of **v**. It is clearly that the direction cosines satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$



Addition of vectors: If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} = \mathbf{b} + \mathbf{a}$$

$$c\mathbf{a} = ca_1\mathbf{i} + ca_2\mathbf{j} + ca_3\mathbf{k}$$

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II. Inner product (Dot product or Scalar product)

Definition: $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta$,

where θ, $0\le\theta\le\pi$, is the angle between **a** and **b**. In components

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$$

Properties: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

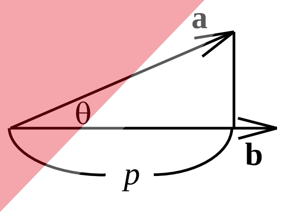
$$\mathbf{a}\cdot\mathbf{a}=a^2$$

Theorem: The inner product of two nonzero vectors is zero if and only if the two vectors are perpendicular.

Projection of a vector **a** in the direction of a vector **b** is

$$p=a\cos\theta=a\frac{\mathbf{a}\cdot\mathbf{b}}{ab}=\frac{\mathbf{a}\cdot\mathbf{b}}{b}=\mathbf{a}\cdot\frac{\mathbf{b}}{b}=\mathbf{a}\cdot\mathbf{e}_b,$$

where $\mathbf{e}_b$ is the unit vector in the direction of **b**.



Ex. 1.

An orthonormal basis for 3-space is $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$, $a=b=c=1$, any vector **v** in the space can be written as

$$\mathbf{v}=(\mathbf{v}\cdot\mathbf{a})\mathbf{a}+(\mathbf{v}\cdot\mathbf{b})\mathbf{b}+(\mathbf{v}\cdot\mathbf{c})\mathbf{c}.$$

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Ex. 2.

Find the straight line which passes through the two points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$.

Solution: Assume any point in the plane is $X(x, y, z)$, then

$$\overrightarrow{PX}=t\overrightarrow{PQ}\Rightarrow[(x-x_1)\mathbf{i}+(y-y_1)\mathbf{j}+(z-z_1)\mathbf{k}]=t[(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}]$$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ is the equation of the line.

Ex. 3.

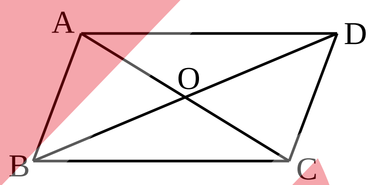
Find a plane which passes through the point P(x_0, y_0, z_0) and is normal to the vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Solution: Assume any point in the plane is X(x, y, z), $\because \overrightarrow{PX} \perp \mathbf{N}$, $\therefore \mathbf{N} \cdot \overrightarrow{PX} = 0$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the equation of the plane.

Ex. 4.



Prove that the diagonals of a parallelogram bisect each other.

Solution: Let $\overline{BC} = \mathbf{a}$, $\overline{BA} = \mathbf{b}$, then

$$\overrightarrow{BO} = m\overrightarrow{BD} = m(\mathbf{a} + \mathbf{b}), \text{ and}$$

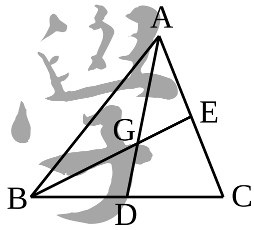
$$\overrightarrow{BO} = \overrightarrow{BC} + \overrightarrow{CO} = \overrightarrow{BC} + n\overrightarrow{CA} = \overrightarrow{BC} + n(\overrightarrow{BA} - \overrightarrow{BC}) = (1 - n)\mathbf{a} + n\mathbf{b}$$

$$m(\mathbf{a} + \mathbf{b}) = (1 - n)\mathbf{a} + n\mathbf{b} \Rightarrow (m + n - 1)\mathbf{a} + (m - n)\mathbf{b} = 0$$

Since $\mathbf{a}$ and $\mathbf{b}$ are independent, $m + n - 1 = 0$, and $m - n = 0$. We get $m = n = 1/2$, and point O is the midpoint of the diagonals.

Ex. 5.

If $\overline{BD} = \overline{CD}$, $\overline{AE} = \overline{CE}$, Prove that $\overline{AG} = \frac{2}{3}\overline{AD}$.



Solution : $\overrightarrow{AG} = t\overrightarrow{AD} = t(\overrightarrow{AB} + \overrightarrow{BD}) = t(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}) = t[\overrightarrow{AB} + \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB})]$

$$= t(\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}) = \frac{t}{2}\overrightarrow{AB} + \frac{t}{2}\overrightarrow{AC}$$

$$\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BG} = \overrightarrow{AB} + s\overrightarrow{BE} = \overrightarrow{AB} + s(\overrightarrow{AE} - \overrightarrow{AB})$$

$$= (1 - s)\overrightarrow{AB} + \frac{s}{2}\overrightarrow{AC}$$

$$\begin{cases} \frac{t}{2} = 1 - s \\ \frac{t}{2} = \frac{s}{2} \end{cases} \Rightarrow t = s = \frac{2}{3} \Rightarrow \overline{AG} = \frac{2}{3}\overline{AD}$$

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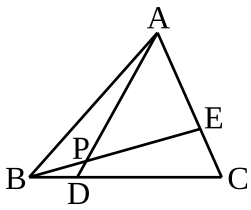
Ex. 6.

If $\overline{CD}=3\overline{BD}$, $\overline{AE}=2\overline{CE}$, Find $\overline{AP}:\overline{PD}$.



Solution : $\overrightarrow{AP}=t\overrightarrow{AD}=t(\overrightarrow{AB}+\overrightarrow{BD})=t(\overrightarrow{AB}+\frac{1}{4}\overrightarrow{BC})=t[\overrightarrow{AB}+\frac{1}{4}(\overrightarrow{AC}-\overrightarrow{AB})]$

$$=t(\frac{3}{4}\overrightarrow{AB}+\frac{1}{4}\overrightarrow{AC})=\frac{3t}{4}\overrightarrow{AB}+\frac{t}{4}\overrightarrow{AC}$$

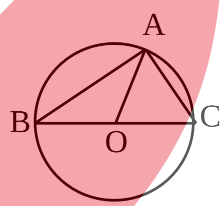
$$\overrightarrow{AP}=\overrightarrow{AB}+\overrightarrow{BP}=\overrightarrow{AB}+s\overrightarrow{BE}=\overrightarrow{AB}+s(\overrightarrow{AE}-\overrightarrow{AB})$$

$$=(1-s)\overrightarrow{AB}+\frac{2s}{3}\overrightarrow{AC}$$

$$\begin{cases}\frac{3t}{4}=1-s\\ \frac{t}{4}=\frac{2s}{3}\end{cases}\Rightarrow t=\frac{8}{9}\Rightarrow \overline{AP}:\overline{PD}=8:1$$

Ex. 7.

Prove that the triangle inscribed in a semicircle is a right triangle.



Solution: Let $\overrightarrow{OC}=\mathbf{c}, \overrightarrow{OA}=\mathbf{a}$, then

$\overrightarrow{BA}=\overrightarrow{BO}+\overrightarrow{OA}=\overrightarrow{OC}+\overrightarrow{OA}=\mathbf{a}+\mathbf{c}$

$\overrightarrow{AC}=\overrightarrow{AO}+\overrightarrow{OC}=\mathbf{c}-\mathbf{a}$

$\overrightarrow{BA}\cdot\overrightarrow{AC}=(\mathbf{a}+\mathbf{c})\cdot(\mathbf{c}-\mathbf{a})=c^2-a^2=0\Rightarrow \overline{AB}$ and $\overline{AC}$ are perpendicular.

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[Exercises] 1. Find all vectors $\mathbf{v}$ such that $(\mathbf{v}\cdot\mathbf{v})\mathbf{v}=169\mathbf{v}$ and $4\mathbf{i}\cdot\mathbf{v}=3\mathbf{j}\cdot\mathbf{v}=\mathbf{k}\cdot\mathbf{v}$.

2. Find the angle between the planes $2x-y+2z=1$ and $x-y=2$.

3. Find the scalar and vector projections of $-7\mathbf{i}+14\mathbf{j}+7\mathbf{k}$ along $2\mathbf{i}+3\mathbf{j}-6\mathbf{k}$.

4. Find the magnitude of the scalar component of the force vector $\mathbf{F}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ in the direction of the straight line with equations $x=y=2z$.

[Answers] 1. $\mathbf{v}=\pm3\mathbf{i}\pm4\mathbf{j}\pm12\mathbf{k}$ 2. $45°$ 3. -2, $(-4\mathbf{i}-6\mathbf{j}+12\mathbf{k})/7$ 4. $8/3$

III. Outer product (Cross product, Vector product)

Definition: The outer product of two vectors **a** and **b** is a vector

$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

The magnitude of **v** is $v = ab\sin\theta$, $0 \le \theta \le \pi$, θ is the angle between **a** and **b**. The direction of **v** is the direction of advance of a right-hand screw as vector **a** is turned into vector **b**, and it is perpendicular to both **a** and **b**. In components, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$\because \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$

$\mathbf{i} \times \mathbf{j} = \mathbf{k},\quad \mathbf{j} \times \mathbf{k} = \mathbf{i},\quad \mathbf{k} \times \mathbf{i} = \mathbf{j},$

$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\quad \mathbf{k} \times \mathbf{j} = -\mathbf{i},\quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

$\therefore v_1 = a_2b_3 - a_3b_2,\ v_2 = a_3b_1 - a_1b_3,\ v_3 = a_1b_2 - a_2b_1$

or $$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

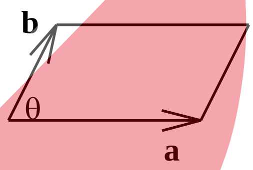
Properties: 1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \ne \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ in general

3. If $\mathbf{a} \times \mathbf{b} = 0$, (1) then one of the two vectors is zero.

(2) **a** and **b** has the same or opposite direction.

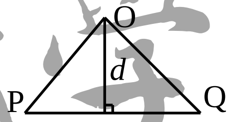
4. $|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$ is the area of the parallelogram formed by the vectors **a** and **b**, as shown.



Ex. 8.

Find the distance d of point O to the line $\overleftrightarrow{PQ}$.

Solution: $d = \dfrac{|\overrightarrow{OP} \times \overrightarrow{OQ}|}{|\overrightarrow{PQ}|}$



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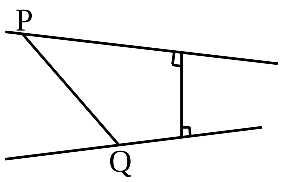
Ex. 9.

Find the distance d between two lines which is not at the same plane.

Solution: Suppose the unit vectors along the two lines are $\mathbf{e}_1$, $\mathbf{e}_2$ respectively, and P, Q are points on the two lines respectively, then

$$d = \left| \overrightarrow{PQ} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} \right|$$



Ex. 10.

Find the distance of two lines L_1: $\dfrac{x-11}{4}=\dfrac{y+5}{-3}=\dfrac{z+7}{-1}$, $L_2:\dfrac{x+5}{3}=\dfrac{y-4}{-4}=\dfrac{z-6}{-2}$.

Solution: (1)L_1的方向向量為 $\mathbf{v}_1=4\mathbf{i}-3\mathbf{j}-\mathbf{k}$，$L_2$的方向向量為 $\mathbf{v}_2=3\mathbf{i}-4\mathbf{j}-2\mathbf{k}$

設 $P(4m+11,-3m-5,-m-7)$為 L_1上的點，

$Q(3n-5,-4n+4,-2n+6)$為 L_2上的點，且 P 與 Q 的距離為二歪斜線的距離，則

$\overrightarrow{PQ}\cdot\mathbf{v}_1=0\Rightarrow[(3n-4m-16)\mathbf{i}+(-4n+3m+9)\mathbf{j}+(-2n+m+13)\mathbf{k}]\cdot(4\mathbf{i}-3\mathbf{j}-\mathbf{k})=0$

$4(3n-4m-16)-3(-4n+3m+9)-(-2n+m+13)=0\Rightarrow 26n-26m=104$……①

$\overrightarrow{PQ}\cdot\mathbf{v}_2=0\Rightarrow[(3n-4m-16)\mathbf{i}+(-4n+3m+9)\mathbf{j}+(-2n+m+13)\mathbf{k}]\cdot(3\mathbf{i}-4\mathbf{j}-2\mathbf{k})=0$

$3(3n-4m-16)-4(-4n+3m+9)-2(-2n+m+13)=0\Rightarrow 26n-29m=110$……②

①－②: $3m=-6\Rightarrow m=-2$，代入①得 $n=2$

m, n 代回得 $P(3, 1, -5)$, $Q(1, -4, 2)$

二歪斜線的距離為 $\overline{PQ}=\sqrt{(-2)^2+(-5)^2+7^2}=\sqrt{78}$

(2)設包含 L_1且平行 L_2的平面 E 為 $a(x-11)+b(y+5)+c(z+7)=0$，則

E 的法向量$\perp L_1\Rightarrow(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(4\mathbf{i}-3\mathbf{j}-\mathbf{k})=0\Rightarrow 4a-3b-c=0$

E 的法向量$\perp L_2\Rightarrow(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}-2\mathbf{k})=0\Rightarrow 3a-4b-2c=0$

$$a:b:c=\begin{vmatrix}-3 & -1\\ -4 & -2\end{vmatrix}:\begin{vmatrix}-1 & 4\\ -2 & 3\end{vmatrix}:\begin{vmatrix}4 & -3\\ 3 & -4\end{vmatrix}=2:5:(-7)$$

平面 E 為 $2(x-11)+5(y+5)-7(z+7)=0\Rightarrow 2x+5y-7z-46=0$

L_2過點 $A(-5, 4, 6)$，則 A 到平面 E 的距離即為兩歪斜線的距離，其值為

$$\frac{|2(-5)+5\times4-7\times6-46|}{\sqrt{2^2+5^2+(-7)^2}}=\frac{78}{\sqrt{78}}=\sqrt{78}$$

(3)L_1的方向向量為 $\mathbf{v}_1=4\mathbf{i}-3\mathbf{j}-\mathbf{k}$，$L_2$的方向向量為 $\mathbf{v}_2=3\mathbf{i}-4\mathbf{j}-2\mathbf{k}$

$A(11, -5, -7)$, $B(-5, 4, 6)$分別在直線 L_1, L_2上

同時與 $\mathbf{v}_1$, $\mathbf{v}_2$垂直的向量為

$$\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -3 & -1\\ 3 & -4 & -2\end{vmatrix}=2\mathbf{i}+5\mathbf{j}-7\mathbf{k}$$

L_1, L_2的距離為 $\left|\overrightarrow{AB}\cdot\dfrac{\mathbf{v}_1\times\mathbf{v}_2}{|\mathbf{v}_1\times\mathbf{v}_2|}\right|=\left|(-16\mathbf{i}+9\mathbf{j}+13\mathbf{k})\cdot\dfrac{2\mathbf{i}+5\mathbf{j}-7\mathbf{k}}{\sqrt{2^2+5^2+(-7)^2}}\right|$

$$=\frac{|-32+45-91|}{\sqrt{78}}=\sqrt{78}$$

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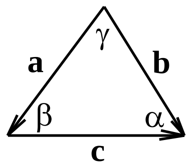
Ex. 11.

Derive the law of sines using vectors.

Solution: $\mathbf{c}=\mathbf{b}-\mathbf{a}\Rightarrow\mathbf{c}\times\mathbf{c}=\mathbf{c}\times(\mathbf{b}-\mathbf{a})\Rightarrow 0=\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}\Rightarrow\mathbf{c}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

$$cb\sin\alpha=ca\sin(\pi-\beta)\Rightarrow\frac{a}{\sin\alpha}=\frac{b}{\sin\beta},\quad \text{Similarly},\ \frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$



Ex. 12.

Find the plane which contains three points A, B and C such that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

Solution: Let $X=(x, y, z)$, then

$$\overrightarrow{AX}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=0$$

where we can find the equation of the plane.

[Exercises] 1. Find a unit vector perpendicular to both $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $-5\mathbf{i}+4\mathbf{j}-2\mathbf{k}$.

2. Find the equation of the line containing (1, 4, 3) which is perpendicular to both of the lines $\frac{x-1}{2}=y+3=\frac{z-2}{4}$ and $\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$.

3. Find parametric equations for the intersection of the planes $2x-y+z=-2$ and $x+y+z=0$.

4. Write the equation of the plane containing the lines $x=y=\frac{4-z}{4}$ and $2x=2-y=z$.

[Answers] 1. $\pm(\mathbf{j}+2\mathbf{k})/\sqrt{5}$ 2. $\frac{x-1}{-10}=\frac{y-4}{16}=z-3$ 3. $x=2t,\ y=1+t,\ z=-1-3t$

4. $2x+2y+z=4$

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IV. Multiple products

1. Scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

Since $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane of **a** and **b**, having a magnitude equal to the area of the parallelogram of which **a** and **b** form coterminous sides, and the projection of **c** on $\mathbf{a}\times\mathbf{b}$ is the altitude of the parallelepiped with **a**, **b**, and **c** as coterminous edges, it follows that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is equal to the volume of this parallelepiped. From the fact it follows easily that

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$$

Since $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, we get $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. It follows that the dot and cross can be interchanged in a scalar triple product, the notation $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ is frequently used to indicate the common value of the products, and in components

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. Vector triple product $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

Since $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is perpendicular to $\mathbf{a}\times\mathbf{b}$, which is itself perpendicular to the plane of **a** and **b**, and also perpendicular to **c**, it follows that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ lies in the plane of **a** and **b** and perpendicular to **c**. Thus

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=m\mathbf{a}+n\mathbf{b}\Rightarrow 0=m\mathbf{a}\cdot\mathbf{c}+n\mathbf{b}\cdot\mathbf{c}$$

If we write $n=\lambda\mathbf{a}\cdot\mathbf{c}$, then $m=-\lambda\mathbf{b}\cdot\mathbf{c}$, and

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\lambda[(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}] \qquad (1.1)$$

To determine λ, let **u** be a unit vector parallel to **a**, **v** be a second unit vector perpendicular to **a** such that **b** is in the plane of **u** and **v**, and $\mathbf{w}=\mathbf{u}\times\mathbf{v}$, then

$$\mathbf{a}=a_1\mathbf{u},\quad \mathbf{b}=b_1\mathbf{u}+b_2\mathbf{v},\quad \mathbf{c}=c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}$$

Substitute into (1.1), there follows

$$[a_1\mathbf{u}\times(b_1\mathbf{u}+b_2\mathbf{v})]\times(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})=\lambda\{[a_1\mathbf{u}\cdot(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})](b_1\mathbf{u}+b_2\mathbf{v})-[(b_1\mathbf{u}+b_2\mathbf{v})\cdot(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})](a_1\mathbf{u})\}$$

$$a_1b_2\mathbf{w}\times(c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w})=\lambda[a_1c_1(b_1\mathbf{u}+b_2\mathbf{v})-(b_1c_1+b_2c_2)(a_1\mathbf{u})]$$

$$a_1b_2c_1\mathbf{v}-a_1b_2c_2\mathbf{u}=\lambda(a_1b_2c_1\mathbf{v}-a_1b_2c_2\mathbf{u})$$

where we can get $\lambda=1$

and

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

Similarly

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

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[Note] $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

3. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ (1.2)

Let $\mathbf{u}=\mathbf{c}\times\mathbf{d}$, (1.2) becomes

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{u}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{u})=\mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]=\mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}]$$
$$=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

4. $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$

Let $\mathbf{u}=\mathbf{c}\times\mathbf{d}$,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\times\mathbf{b})\times\mathbf{u}=(\mathbf{a}\cdot\mathbf{u})\mathbf{b}-(\mathbf{b}\cdot\mathbf{u})\mathbf{a}=[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\mathbf{b}-[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\mathbf{a}$$
$$=(\mathbf{a}\ \mathbf{c}\ \mathbf{d})\mathbf{b}-(\mathbf{b}\ \mathbf{c}\ \mathbf{d})\mathbf{a}$$

If we let $\mathbf{v}=\mathbf{a}\times\mathbf{b}$,

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{v}\times(\mathbf{c}\times\mathbf{d})=(\mathbf{v}\cdot\mathbf{d})\mathbf{c}-(\mathbf{v}\cdot\mathbf{c})\mathbf{d}=[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\mathbf{c}-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\mathbf{d}$$
$$=(\mathbf{a}\ \mathbf{b}\ \mathbf{d})\mathbf{c}-(\mathbf{a}\ \mathbf{b}\ \mathbf{c})\mathbf{d}$$

[Exercise] Derive the identity $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})=(\mathbf{A}\ \mathbf{B}\ \mathbf{C})^2$

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