Chapter 3 Fourier Integral

I. Fourier Integral

We have described Fourier series for functions which are periodic. However, functions which are not periodic cannot be represented by a Fourier series. In many problems it is desirable to develop an integral representation for such a function that is analogous to a Fourier series. The Fourier series for $f(x)$ in the interval $[-\mathrm{L}, \mathrm{L}]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{\mathrm{L}}x + b_n \sin\frac{n\pi}{\mathrm{L}}x\right) \qquad (3.1)$$

where

$$a_n = \frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\cos\frac{n\pi u}{\mathrm{L}}du \qquad n = 0, 1, 2, 3, \cdots$$

$$b_n = \frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\sin\frac{n\pi u}{\mathrm{L}}du \qquad n = 1, 2, 3, \cdots$$

Substituting into (3.1), we obtain

$$f(x) = \frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)du + \sum_{n=1}^{\infty}\frac{1}{\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\left(\cos\frac{n\pi u}{\mathrm{L}}\cos\frac{n\pi x}{\mathrm{L}} + \sin\frac{n\pi u}{\mathrm{L}}\sin\frac{n\pi x}{\mathrm{L}}\right)du$$

$$f(x) = \frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)du + \frac{1}{\mathrm{L}}\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\cos\frac{n\pi(u-x)}{\mathrm{L}}du \qquad (3.2)$$

Suppose now that $f(x)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty}|f(u)|du$$

converges. Then

$$\frac{a_0}{2} = \frac{1}{2\mathrm{L}}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)du \le \frac{1}{2\mathrm{L}}\left|\int_{-\mathrm{L}}^{\mathrm{L}} f(u)du\right| \le \frac{1}{2\mathrm{L}}\int_{-\infty}^{\infty}|f(u)|du$$

which approaches zero as $\mathrm{L} \to \infty$. Thus, holding x fixed, as L approach infinity, equation (3.2) becomes

$$f(x) = \lim_{\mathrm{L}\to\infty}\frac{1}{\mathrm{L}}\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\cos\frac{n\pi(u-x)}{\mathrm{L}}du \qquad (3.3)$$

Now let $\omega_n = \frac{n\pi}{\mathrm{L}}$, $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{\mathrm{L}}$, (3.3) becomes

$$f(x) = \frac{1}{\pi}\lim_{\mathrm{L}\to\infty}\Delta\omega\sum_{n=1}^{\infty}\int_{-\mathrm{L}}^{\mathrm{L}} f(u)\cos\omega_n(u-x)du = \frac{1}{\pi}\int_0^{\infty}d\omega\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos\omega(u-x)du\,d\omega \qquad (3.4)$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}[f(u)\cos\omega u\cos\omega x + f(u)\sin\omega u\sin\omega x]du\,d\omega$$

Hence, we write the above equation as

$$f(x) = \int_0^{\infty}[a(\omega)\cos\omega x + b(\omega)\sin\omega x]d\omega \qquad (3.5)$$

where

$$a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx$$

$$b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx$$

Equation (3.5) is the Fourier integral representation for the function $f(x)$.

(1) If $f(x)$ is even, $b(\omega)=0$, equation (3.5) reduces to

$$f(x)=\int_{0}^{\infty} a(\omega)\cos\omega x d\omega$$

$$a(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(x)\cos\omega x dx \qquad (3.6)$$

equation (3.6) is called the Fourier cosine integral for $f(x)$

(2) If $f(x)$ is odd, $a(\omega)=0$, equation (3.5) reduces to

$$f(x)=\int_{0}^{\infty} b(\omega)\sin\omega x d\omega$$

$$b(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(x)\sin\omega x dx \qquad (3.7)$$

equation (3.7) is called the Fourier sine integral for $f(x)$

(3) Since $\cos\omega(u-x)=\frac{1}{2}[e^{i\omega(u-x)}+e^{-i\omega(u-x)}]$, equation (3.4) becomes

$$\begin{aligned}
f(x)&=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)[e^{i\omega(u-x)}+e^{-i\omega(u-x)}]dud\omega\\
&=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\omega(u-x)}dud\omega+\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega\\
&=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i(-\omega)(u-x)}dud(-\omega)+\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega\\
&=\frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega+\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\right]e^{i\omega x}d\omega
\end{aligned}$$

We can write

$$f(x)=\int_{-\infty}^{\infty} c(\omega)e^{i\omega x}d\omega \qquad (3.8)$$

where

$$c(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

equation (3.8) is called the complex form of the Fourier integral for $f(x)$.

[Fourier integral theorem] If $\int_{-\infty}^{\infty}|f(x)|dx$ exists and $f(x)$ is piecewise smooth, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u-x)}dud\omega=\frac{f(x^-)+f(x^+)}{2}$$

Ex. 1

(1) Find the Fourier integral representation for the function $f(x)=\begin{cases}1, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1\end{cases}$, and

(2) Prove $\displaystyle\int_0^\infty \frac{\cos\omega x \sin\omega\, d\omega}{\omega} = \begin{cases}\frac{\pi}{2}, & \text{if } 0\le x<1 \\ \frac{\pi}{4}, & \text{if } x=1 \\ 0, & \text{if } x>1\end{cases}$.

Solution : (1) Let $f(x)=\displaystyle\int_0^\infty [a(\omega)\cos\omega x + b(\omega)\sin\omega x]d\omega$

$$a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx = \frac{1}{\pi}\int_{-1}^{1}\cos\omega x dx = \frac{1}{\pi\omega}[\sin(\omega)-\sin(-\omega)] = \frac{2\sin\omega}{\pi\omega}$$

$$b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx = 0$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega$$

(2) When $0\le x<1$, $\displaystyle\frac{f(x^-)+f(x^+)}{2} = \frac{1+1}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega$

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega = \frac{\pi}{2}$$

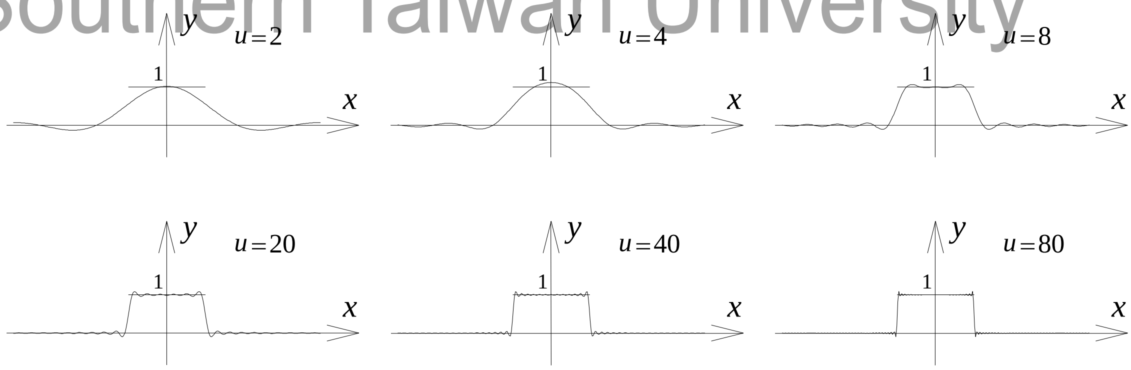
When $x=1$, $\displaystyle\frac{f(x^-)+f(x^+)}{2} = \frac{1+0}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega$

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega = \frac{\pi}{4}$$

When $x>1$, $\displaystyle\frac{f(x^-)+f(x^+)}{2} = \frac{0+0}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega \Rightarrow \int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x d\omega = 0$

Hence, $\displaystyle\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega} d\omega = \begin{cases}\frac{\pi}{2}, & \text{if } 0\le x<1 \\ \frac{\pi}{4}, & \text{if } x=1 \\ 0, & \text{if } x>1\end{cases}$

The graph of $\displaystyle\frac{2}{\pi}\int_0^u \frac{\cos\omega x \sin\omega}{\omega} d\omega$ is shown as follows:



Ex. 2

If $f(x)=\begin{cases}\sin x, & 0\le x\le\pi\\ 0, & \text{otherwise}\end{cases}$, (1)find the Fourier integral representation of $f(x)$, and (2)evaluate $\int_0^\infty \frac{\cos(\omega\pi/2)}{1-\omega^2}d\omega$.

Solution: (1) The Fourier integral representation of $f(x)$ is

$$f(x)=\int_0^\infty [a(\omega)\cos\omega x+b(\omega)\sin\omega x]d\omega$$

where

$$a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx=\frac{1}{\pi}\int_0^\pi \sin x\cos\omega x dx$$

$$=\frac{1}{2\pi}\int_0^\pi[\sin(1+\omega)x+\sin(1-\omega)x]dx=-\frac{1}{2\pi}\left[\frac{\cos(1+\omega)x}{1+\omega}+\frac{\cos(1-\omega)x}{1-\omega}\right]_0^\pi$$

$$=-\frac{1}{2\pi}\left[\frac{\cos(1+\omega)\pi-1}{1+\omega}+\frac{\cos(1-\omega)\pi-1}{1-\omega}\right]$$

$$=-\frac{1}{2\pi}\frac{(1-\omega)[\cos(1+\omega)\pi-1]+(1+\omega)[\cos(1-\omega)\pi-1]}{1-\omega^2}$$

$$=-\frac{1}{2\pi}\frac{[\cos(1+\omega)\pi+\cos(1-\omega)\pi]+\omega[\cos(1-\omega)\pi-\cos(1+\omega)\pi]-2}{1-\omega^2}$$

$$=-\frac{1}{2\pi}\frac{2\cos\pi\cos\omega\pi+\omega\cdot 2\sin\pi\sin\omega\pi-2}{1-\omega^2}=\frac{\cos\omega\pi+1}{\pi(1-\omega^2)}$$

$$b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx=\frac{1}{\pi}\int_0^\pi \sin x\sin\omega x dx$$

$$=\frac{1}{2\pi}\int_0^\pi[\cos(1-\omega)x-\cos(1+\omega)x]dx=\frac{1}{2\pi}\left[\frac{\sin(1-\omega)x}{1-\omega}-\frac{\sin(1+\omega)x}{1+\omega}\right]_0^\pi$$

$$=\frac{1}{2\pi}\left[\frac{\sin(1-\omega)\pi-0}{1-\omega}-\frac{\sin(1+\omega)\pi-0}{1+\omega}\right]$$

$$=\frac{1}{2\pi}\cdot\frac{(1+\omega)\sin(1-\omega)\pi-(1-\omega)\sin(1+\omega)\pi}{1-\omega^2}$$

$$=\frac{1}{2\pi}\frac{[\sin(1-\omega)\pi-\sin(1+\omega)\pi]+\omega[\sin(1-\omega)\pi+\sin(1+\omega)\pi]}{1-\omega^2}$$

$$=\frac{1}{2\pi}\frac{-2\cos\pi\sin\omega\pi+\omega\cdot 2\sin\pi\sin\omega\pi}{1-\omega^2}=\frac{\sin\omega\pi}{\pi(1-\omega^2)}$$

hence

$$f(x)=\frac{1}{\pi}\int_0^\infty\left(\frac{\cos\omega\pi+1}{1-\omega^2}\cos\omega x+\frac{\sin\omega\pi}{1-\omega^2}\sin\omega x\right)d\omega=\frac{1}{\pi}\int_0^\infty\frac{\cos\omega x+\cos\omega(\pi-x)}{1-\omega^2}d\omega$$

(2) Let $x=\pi/2$, we have

$$\frac{f(\pi^-/2)+f(\pi^+/2)}{2}=\frac{1}{\pi}\int_0^\infty\frac{2\cos(\omega\pi/2)}{1-\omega^2}d\omega\Rightarrow 1=\frac{2}{\pi}\int_0^\infty\frac{\cos(\omega\pi/2)}{1-\omega^2}d\omega$$

$$\int_0^\infty\frac{\cos(\omega\pi/2)}{1-\omega^2}d\omega=\frac{\pi}{2}$$

[Exercise 1] (1)Find the Fourier integral representation of $f(x)=\begin{cases} 0, & \text{if } -\infty < x \le -1 \\ 1+x, & \text{if } -1 < x \le 0 \\ 1-x, & \text{if } 0 < x \le 1 \\ 0, & \text{if } 1 < x < \infty \end{cases}$,

(2)and then determine the value of $\int_0^\infty \frac{1-\cos\omega}{\omega^2}\,d\omega$.

[Exercise 2] If $f(x)=\begin{cases} 1 & \text{if } 0 < x \le a \\ \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x > a \end{cases}$, (1)find the Fourier cosine integral of $f(x)$;

(2)evaluate the integral $\int_0^\infty \frac{\sin 2ax}{x}\,dx$.

[Solution](1) Let $f(x) = \int_0^\infty a(\omega)\cos\omega x\,d\omega$

$$a(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = \frac{2}{\pi}\int_0^a \cos\omega x\,dx = \frac{2\sin a\omega}{\pi\omega}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\omega}{\omega}\cos\omega x\,d\omega$$

(2) Substitute $x = a$ into $f(x)$

$$\frac{f(a^-)+f(a^+)}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin a\omega}{\omega}\cos\omega a\,d\omega \Rightarrow \frac{1+0}{2} = \frac{1}{\pi}\int_0^\infty \frac{\sin 2a\omega}{\omega}\,d\omega$$

$$\int_0^\infty \frac{\sin 2a\omega}{\omega}\,d\omega = \frac{\pi}{2}$$

[Ans.] 1. (1) $f(x)=\frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega}{\omega^2}\cos\omega x\,d\omega$ (2)$\pi/2$

2. (1) $f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega a}{\omega}\cos\omega x\,d\omega$ (2)$\pi/2$

II. Fourier Transform

1. Definition of Fourier transform

From equation (3.8), if we define

$$\mathcal{F}\,[f(x)] \equiv F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

then

$$\mathcal{F}^{-1}[F(\omega)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

$F(\omega)$ is called the Fourier transform of $f(x)$.

Similarly from equation (3.6), if

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x dx$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos \omega x d\omega$$

$F_c(\omega)$ is called the Fourier cosine transform of $f(x)$.

From equation (3.7), if

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x dx$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega x d\omega$$

$F_s(\omega)$ is called the Fourier sine transform of $f(x)$.

III. Properties of Fourier Transform

Let

$$F(\omega) = \mathrm{F}\,[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

and

$$f(t) = \mathrm{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

$F_1(\omega) = \mathrm{F}\,[f_1(t)]$, $F_2(\omega) = \mathrm{F}\,[f_2(t)]$

$f(t)*g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$ is defined as the convolution of $f(t)$ and $g(t)$.

Theorem	Original function of time t	Transformed function of frequency ω
Linearity	$c_1f_1(t) + c_2f_2(t)$	$c_1F_1(\omega) + c_2F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a \rvert}F(\frac{\omega}{a})$
Symmetry	$F(t)$	$2\pi f(-\omega)$
Conjugate	$f^*(t)$	$F^*(-\omega)$
Shifting	$f(t-t_0)$	$F(\omega)\,e^{-i\omega t_0}$
	$f(t)\,e^{i\omega_0 t}$	$F(\omega-\omega_0)$
Differentiation	$\frac{d^n f(t)}{dt^n}$	$(i\omega)^n F(\omega)$
	$(-it)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Parseval	$\int_{-\infty}^{\infty}\lvert f(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F(\omega)\rvert^2\,d\omega$	

1. Linearity

$$\mathrm{F}\,[c_1f_1(t) + c_2f_2(t)] = \int_{-\infty}^{\infty}[c_1 f_1(t) + c_2 f_2(t)]e^{-i\omega t}dt = c_1\int_{-\infty}^{\infty} f_1(t)e^{-i\omega t}dt + c_2\int_{-\infty}^{\infty} f_2(t)e^{-i\omega t}dt$$

$$= c_1\mathrm{F}\,[f_1(t)] + c_2\mathrm{F}\,[f_2(t)] = c_1F_1(\omega) + c_2F_2(\omega)$$

2. Scaling

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt$$, Let $at = u$, we have

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(u)e^{-i\omega\frac{u}{a}}d\frac{u}{a} = \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)e^{-i\frac{\omega}{a}u}du = \frac{1}{|a|}F(\frac{\omega}{a})$$

3. Symmetry

Since

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(-t)}d\omega$$

interchange t and ω, we have

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)e^{-it\omega}dt = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt$$

then $2\pi f(-\omega)$ is the Fourier transform of $F(t)$, i.e.,

$$\mathcal{F}[F(t)] = 2\pi f(-\omega)$$

4. Conjugate

Suppose the time function is complex, i.e., $f(t) = x(t) + iy(t)$, its conjugate is $f^*(t) = x(t) - iy(t)$, then

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty}[x(t) + iy(t)]e^{-i\omega t}dt = F(\omega)$$

and

$$\mathcal{F}[f^*(t)] = \int_{-\infty}^{\infty}[x(t) - iy(t)]e^{-i\omega t}dt = \left[\int_{-\infty}^{\infty}[x(t) + iy(t)]e^{i\omega t}dt\right]^* = [F(-\omega)]^* = F^*(-\omega)$$

If $f(t)$ is real, then

$$f^*(t) = f(t), \Rightarrow \mathcal{F}[f^*(t)] = \mathcal{F}[f(t)]$$

$$F^*(-\omega) = F(\omega), \Rightarrow F(-\omega) = F^*(\omega)$$

5. Shifting

(1) Shifting of time axis

$$\mathcal{F}[f(t-a)] = \int_{-\infty}^{\infty} f(t-a)e^{-i\omega t}dt$$

Let $u = t-a$, the right hand side becomes

$$\int_{-\infty}^{\infty} f(u)e^{-i\omega(u+a)}du = e^{-i\omega a}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du = e^{-i\omega a}F[f(t)] = F(\omega)e^{-i\omega a}$$

$$\Rightarrow \mathcal{F}[f(t-a)] = F(\omega)e^{-i\omega a}$$

(2) Shifting of frequency axis

$$\mathcal{F}^{-1}[F(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega-\omega_0)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{i(\xi+\omega_0)t}d\xi$$

$$= \frac{1}{2\pi}e^{i\omega_0 t}\int_{-\infty}^{\infty} F(\xi)e^{i\xi t}d\xi = f(t)\,e^{i\omega_0 t}$$

6. Differentiation

(1) Time differentiation

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = (i\omega)\mathcal{F}[f(t)] = (i\omega)F(\omega)$$

Similarly,

$$\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$$

(2) Frequency differentiation

$$\frac{d}{d\omega}F(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}(-it)f(t)e^{-i\omega t}dt = \mathcal{F}[(-it)f(t)]$$

Similarly,

$$\frac{d^n}{d\omega^n}F(\omega) = \mathcal{F}[(-it)^n f(t)]$$

7. Convolution

(1) Time convolution

$$\mathcal{F}[f_1(t)*f_2(t)]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)d\tau\right]e^{-i\omega t}dt=\int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-i\omega t}dt\right]d\tau$$

From the time-shifting theorem, we conclude that the bracket above equals $F_2(\omega)e^{-i\omega\tau}$, therefore

$$\mathcal{F}[f_1(t)*f_2(t)]=\int_{-\infty}^{\infty} f_1(\tau)e^{-i\omega\tau}F_2(\omega)d\tau=F_1(\omega)F_2(\omega)$$

(2) Frequency convolution

$$f_1(t)\, f_2(t)=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)e^{iut}du\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\omega)e^{i\omega t}d\omega\right]=\frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(u)F_2(\omega)e^{i(u+\omega)t}du\,d\omega$$

Let $u+\omega=v$, $\Rightarrow \omega=v-u$, There follows

$$f_1(t)\, f_2(t)=\frac{1}{2\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(u)F_2(v-u)du\right]e^{ivt}dv=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}F_1(v)*F_2(v)\right]e^{ivt}dv$$

$$\mathcal{F}[f_1(t)\, f_2(t)]=\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$$

8. Parseval's theorem

From the convolution theorem and the conjugate theorem we have

$$\mathcal{F}[f(t)\, f^*(t)]=\frac{1}{2\pi}F(\omega)*F^*(-\omega)$$

$$\int_{-\infty}^{\infty} f(t)\, f^*(t)e^{-i\omega t}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)*F^*(-(\omega-u))du$$

Let $\omega=0$, then

$$\int_{-\infty}^{\infty}|f(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(u)|^2du=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega$$

IV. Some Important Fourier Transform

1. $f(t)=\begin{cases}1, & |t|<a \\ 0, & |t|>a\end{cases} \quad \Leftrightarrow \quad F(\omega)=\dfrac{2\sin a\omega}{\omega}$

2. $f(t)=\begin{cases}1, & 0<t<a \\ 0, & t>a\end{cases} \quad \Leftrightarrow \quad F_C(\omega)=\dfrac{\sin a\omega}{\omega},\quad F_S(\omega)=\dfrac{1-\cos a\omega}{\omega}$

3. $f(t)=e^{-at},\ a>0 \quad \Leftrightarrow \quad F_C(\omega)=\dfrac{a}{\omega^2+a^2},\quad F_S(\omega)=\dfrac{\omega}{\omega^2+a^2}$

Ex. 3

If the Fourier transform of $f(t)$ is $F(\omega)$, find the Fourier transform of $f(t)\cos at$.

Solution: $\mathcal{F}[f(t)\cos at]=\mathcal{F}[f(t)\dfrac{e^{iat}+e^{-iat}}{2}]=\mathcal{F}[\dfrac{f(t)e^{iat}}{2}]+\mathcal{F}[\dfrac{f(t)e^{-iat}}{2}]$

$$=\frac{1}{2}[F(\omega-a)+F(\omega+a)]$$

Ex. 4

$$f(x)=\begin{cases}1, & 0<x<a\\ \frac{1}{2}, & x=a\\ 0, & x>a\end{cases}$$, (a) Find the Fourier cosine transform of $f(x)$, (b) Evaluate $\int_0^\infty \frac{\sin 2ax}{x}dx$.

Solution: (a) $F_c(\omega)=\int_0^\infty f(x)\cos\omega x dx=\int_0^a \cos\omega x dx=\frac{\sin\omega a}{\omega}$

(b) $f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega a}{\omega}\cos\omega x d\omega$

At $x=a$, $\frac{2}{\pi}\int_0^\infty \frac{\sin\omega a}{\omega}\cos\omega a d\omega=\frac{f(a^-)+f(a^+)}{2}=\frac{1+0}{2}=\frac{1}{2}$

$\frac{1}{\pi}\int_0^\infty \frac{\sin 2\omega a}{\omega}d\omega=\frac{1}{2}$, $\Rightarrow \int_0^\infty \frac{\sin 2\omega a}{\omega}d\omega=\frac{\pi}{2}$

Ex. 5

(a) Compute the convolution of $f(x)$, $g(x)$ when $f(x)=g(x)=\begin{cases}1, & -a\le x\le a\\ 0, & |x|>a\end{cases}$

(b) Use the convolution theorem $H(\lambda)=F(\lambda)*G(\lambda)$ and the concept of inverse Fourier transform to evaluate $\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda$

Solution: (a) $g(x-t)=\begin{cases}1, & -a\le x-t\le a\\ 0, & |x-t|>a\end{cases}=\begin{cases}1, & x-a\le t\le x+a\\ 0, & |x-t|>a\end{cases}$

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(t)g(x-t)dt=\int_{-a}^{a} g(x-t)dt=\int_{-a}^{a} u(t-x+a)-u(t-x-a)dt$$

$$=[(t-x+a)u(t-x+a)-(t-x-a)u(t-x-a)]_{-a}^{a}$$

$$=[(2a-x)\,u(2a-x)-(-x)\,u(-x)]-[(-x)\,u(-x)-(-2a-x)\,u(-2a-x)]$$

$$=(2a-x)\,u(2a-x)+2x\,u(-x)-(2a+x)\,u(-2a-x)$$

(b) $\because F(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx=\int_{-a}^{a} e^{-i\omega x}dx=\frac{e^{-i\omega a}-e^{i\omega a}}{-i\omega}=\frac{2\sin\omega a}{\omega}=G(\omega)$

$$f(x)*g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}dx$$

$$(2a-x)u(2a-x)+2xu(-x)-(2a+x)u(-2a-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin\omega a}{\omega}\right)^2 e^{i\omega x}d\omega$$

Let $x=0$, we get

$$2a=\frac{1}{2\pi}\times 4\int_{-\infty}^{\infty}\left(\frac{\sin\omega a}{\omega}\right)^2 d\omega=\frac{2a}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\omega a}{\omega a}\right)^2 d(\omega a)$$

Let $\omega a=\lambda$, then

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda=2,\quad \Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda=\pi$$

Ex. 6

Use Parseval's equation to evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega$.

Solution : Let $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-1}^{1} e^{-i\omega x} dx = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} = \frac{2\sin\omega}{\omega}$$

From Parseval's equation, we have

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \Rightarrow \int_{-1}^{1} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2\sin\omega}{\omega} \right)^2 d\omega$$

$$2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega \quad \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$$