Chapter 3 Ordinary Differential Equations of the First Order and Higher Degree

I. Equations Solvable for y'

Definition: $f(x, y, y')$ can be replaced by a product of factors, rational in x and y, each of lower degree in y', then $f(x, y, y')$ is said to be reducible, otherwise, it is said to be irreducible.

Ex1 $y'^2 + (x+y)y' + xy = 0$

Solution: $(y'+x)(y'+y)=0$

$y'+x=0$ or $y'+y=0$

$y=\frac{x^2}{2}+C$ or $y=Ce^{-x}$

Ex2 $y'^2 + y^2 = 1$

Solution: $y' = \pm\sqrt{1-y^2}$

$y'=\sqrt{1-y^2}$ or $y'=-\sqrt{1-y^2}$

$\sin^{-1}y = x + C$ or $\cos^{-1}y = x + C$

$y=\sin(x+C)$ or $y=\cos(x+C)$

The two solutions are the same.

[Exercises] 1. $xyy'^2 + (x^2 - y^2)y' - xy = 0$

2. $(2xy'-y)^2 = 8x^3$

[Answers] 1. $y=Cx,\ x^2+y^2=C$ 2. $y^2 = 2x(x+C)^2$

II. Equation solvable for $y=f(x, y')$ (3.1)

Differentiating (3.1), we obtain

$$y' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

Since y is no longer present, it may be looked upon as an equation of the first order in the variables x and y'. It may happen that we can integrate this equation. Suppose its solution to be

$$w(x, y', C) = 0 \tag{3.2}$$

Eliminating y' between (3.1) and (3.2), we have the general solution.

Ex3 $4xy'^2 + 2xy' - y = 0$

Solution : Differentiating $\Rightarrow (4y'^2 + 8xy'\frac{dy'}{dx}) + (2y' + 2x\frac{dy'}{dx}) - y' = 0$

$$(4y'^2 + 8xy'\frac{dy'}{dx}) + (y' + 2x\frac{dy'}{dx}) = 0 \Rightarrow 4y'(y' + 2x\frac{dy'}{dx}) + (y' + 2x\frac{dy'}{dx}) = 0$$

$$(4y'+1)(y' + 2x\frac{dy'}{dx}) = 0$$

(1) $y' + 2x\frac{dy'}{dx} = 0 \Rightarrow \frac{dx}{x} + 2\frac{dy'}{y'} = 0 \Rightarrow \ln x + 2\ln y' = C_1 \Rightarrow xy'^2 = C \Rightarrow y' = \pm\sqrt{\frac{C}{x}}$

Substituting into the original equation, we have

$4C \pm 2\sqrt{Cx} - y = 0 \Rightarrow (y - 4C)^2 = 4Cx \Rightarrow (y-k)^2 = kx$

(2) $4y' + 1 = 0 \Rightarrow y' = -1/4$, Substitute into the original equation, there follows

$\frac{x}{4} + \left(-\frac{x}{2}\right) - y = 0 \Rightarrow x + 4y = 0$

This is the singular solution.

Ex4 $y + xy' = x^4y'^2$

Solution : Differentiating $\Rightarrow y' + y' + x\dfrac{dy'}{dx} = 4x^3y'^2 + 2x^4y'\dfrac{dy'}{dx}$

$$(1-2x^3y')(2y' + x\frac{dy'}{dx}) = 0$$

1. $2y' + x\dfrac{dy'}{dx} = 0 \Rightarrow \dfrac{2dx}{x} + \dfrac{dy'}{y'} = 0 \Rightarrow x^2y' = C \Rightarrow y' = \dfrac{C}{x^2}$

Substituting into the original equation, we have

$$y + x(C/x^2) = x^4(C/x^2)^2 \Rightarrow xy = C^2x - C$$

2. $1-2x^3y' = 0 \Rightarrow y' = 1/2x^3$, The singular solution is

$$y + x(1/2x^3) = x^4(1/2x^3)^2 \Rightarrow y + 1/2x^2 = 1/4x^2 \Rightarrow 4x^2y + 1 = 0$$

[Exercises] Find the general solutions of the following equations:

1. $xy'^2 - 2yy' - x = 0$
2. $4x^2y - 2x^3y' + ay'^2 = 0$

[Answers] 1. $x^2 - 2Cy - C^2 = 0$ 2. $y - Cx^2 + aC^2 = 0$

III. Equation solvable for $x = f(y, y')$ (3.3)

Differentiating (3.3) with respect to y, we obtain

$$\frac{dx}{dy} = \frac{1}{y'} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}\frac{dy'}{dy}$$

We may look upon this equation as the first order in y and y'. If we can integrate it, we obtain

$$w(y, y', C) = 0 \quad (3.4)$$

Eliminating y' between (3.3) and (3.4), we get the general solution.

Ex5 $2yy'^2 - 2xy' + y = 0$

Solution : $x=\dfrac{2yy'^2+y}{2y'}\Rightarrow x=yy'+\dfrac{y}{2y'}$

$$\frac{dx}{dy}=(y'+y\frac{dy'}{dy})+(\frac{1}{2y'}-\frac{y}{2y'^2}\frac{dy'}{dy})$$

$$\frac{1}{y'}=(y'+y\frac{dy'}{dy})+(\frac{1}{2y'}-\frac{y}{2y'^2}\frac{dy'}{dy})\Rightarrow 0=(y'+y\frac{dy'}{dy})-(\frac{1}{2y'}+\frac{y}{2y'^2}\frac{dy'}{dy})$$

$$(y'+y\frac{dy'}{dy})-\frac{1}{2y'^2}(y'+y\frac{dy'}{dy})=0\Rightarrow(y'+y\frac{dy'}{dy})(1-\frac{1}{2y'^2})=0$$

(1) $y'+y\dfrac{dy'}{dy}=0\Rightarrow\dfrac{dy}{y}+\dfrac{dy'}{y'}=0\Rightarrow\ln y+\ln y'=C_1\Rightarrow yy'=C\Rightarrow y'=\dfrac{C}{y}$

Substituting into the original equation, we have

$$2y\left(\frac{C}{y}\right)^2-2x\left(\frac{C}{y}\right)+y=0\Rightarrow 2C^2-2Cx+y^2=0$$

(2) $1-\dfrac{1}{2y'^2}=0\Rightarrow y'=\pm\sqrt{\dfrac{1}{2}}$, substituting into the original equation, we have

$$2y\left(\frac{1}{2}\right)-2x\left(\pm\sqrt{\frac{1}{2}}\right)+y=0\Rightarrow y\pm\sqrt{\frac{1}{2}}x=0$$

Ex6 $y^2y'^2-3xy'+y=0$

Solution : $x=\dfrac{y^2y'^2+y}{3y'}\Rightarrow\dfrac{1}{y'}=\dfrac{2yy'^2+1}{3y'}+\left(\dfrac{y^2}{3}-\dfrac{y}{3y'^2}\right)\dfrac{dy'}{dy}$

$$0=\frac{2yy'^2-2}{3y'}+\frac{yy'^2-1}{3y'^2}y\frac{dy'}{dy}\Rightarrow\frac{yy'^2-1}{3}\left(2+\frac{y}{y'}\frac{dy'}{dy}\right)=0$$

1. $2+\dfrac{y}{y'}\dfrac{dy'}{dy}=0\Rightarrow y^2y'=C\Rightarrow y'=\dfrac{C}{y^2}$, the general solution is

$$y^2\left(\frac{C}{y^2}\right)^2-3x\left(\frac{C}{y^2}\right)+y=0\Rightarrow C^2-3Cx+y^3=0$$

2. $\dfrac{yy'^2-1}{3}=0\Rightarrow y'=\pm\sqrt{\dfrac{1}{y}}$, the singular solution is

$$y^2\left(\frac{1}{y}\right)-3x\left(\pm\sqrt{\frac{1}{y}}\right)+y=0\Rightarrow 2y\pm 3x\sqrt{\frac{1}{y}}=0$$

[Exercises] Find the general solutions of the following equations:

1. $a^2yy'^2-2xy'+y=0$

2. $y'^3-4xyy'+8y^2=0$

3. $2yy'^3-3xy'+2y=0$

[Answers] 1. $y^2-2Cy+a^2C^2=0$ 2. $y=C(x-C)^2$ 3. $4y^3=C(3x-2C)^2$

IV. Clairaut equation: $y-xy'=f(y')$ (3.5)

Differentiating (3.5), we get

$$-x\frac{dy'}{dx}=\frac{\partial f}{\partial y'}\frac{dy'}{dx}\Rightarrow\left(x+\frac{\partial f}{\partial y'}\right)\frac{dy'}{dx}=0$$

The vanish of the first term $x+\frac{\partial f}{\partial y'}=0$ deduce the singular solution, the second $\frac{dy'}{dx}=0$, has

$$y'=C$$

Hence the general solution is

$$y-Cx=f(C)$$

Ex7 $xy'^2-yy'+3=0$

Solution: The equation can be written as

$$y-xy'=\frac{3}{y'}$$

This is Clairaut equation, the general solution is

$$C^2x-Cy+3=0$$

Ex8 $x^2y'^2-2(xy-4)y'+y^2=0$

Solution: The equation can be written as

$$x^2y'^2-2xyy'+8y'+y^2=0$$

$$(xy'-y)^2+8y'=0$$

This is Clairaut equation, the general solution is

$$C^2x^2-2C(xy-4)+y^2=0$$