Chapter2 Ordinary Differential Equations of the First Order and First Degree

General form: 1. $M(x,y)dx + N(x,y)dy = 0$ (2.1a)

2. $y' = f(x,y)$ (2.1b)

I. Separable Differential equations

Form: $M_1(x)N_1(y)dx + M_2(x)N_2(y)dy = 0$

$$\int \frac{M_1(x)}{M_2(x)}dx + \int \frac{N_2(y)}{N_1(y)}dy = C$$

Ex1 $9yy' + 4x = 0$

Solution: $9ydy + 4xdx = 0 \Rightarrow 9y^2 + 4x^2 = C$

Ex2 $y' = 1 + y^2$

Solution: $\dfrac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x + C \Rightarrow y = \tan(x+C)$

Ex3 $y' + 5x^4y^2 = 0,\ y(0) = 1$

Solution: $\dfrac{dy}{y^2} = -5x^4dx \Rightarrow -\dfrac{1}{y} = -x^5 + C$

$y(0) = 1 \Rightarrow -1 = C \Rightarrow$ The solution is $\dfrac{1}{y} = x^5 + 1$

Ex4 $y' = -2xy,\ y(0) = 1$

Solution: $\dfrac{dy}{y} = -2xdx \Rightarrow \ln y = -x^2 + C' \Rightarrow y = e^{-x^2+C'} \Rightarrow y = Ce^{-x^2}$

$y(0) = 1 \Rightarrow 1 = Ce^0 \Rightarrow C = 1 \Rightarrow$ The solution is $y = e^{-x^2}$

II. Reducible to separable differential equations

1. Homogeneous equation

Definition: If $f(tx, ty) = t^r f(x, y)$, then $f(x, y)$ is a homogeneous function of degree r.

Substituting $t = 1/x$ into $f(tx, ty) = t^r f(x, y)$, we have

$$f(1, \frac{y}{x}) = \frac{1}{x^r} f(x, y) \Rightarrow f(x, y) = x^r f(1, \frac{y}{x}) \equiv x^r F(\frac{y}{x}).$$

In particular $r = 0$, then $f(x, y) = F(\frac{y}{x})$

If $M(x, y)dx + N(x, y)dy = 0$, when $M(x, y)$ and $N(x, y)$ are of the same degree in x and y, M/N is a homogeneous function of degree zero, and the differential equation can be written as

$$\frac{dy}{dx} = -\frac{M}{N} = F(\frac{y}{x}) \qquad (2.2)$$

Let $y = ux$, equation (2.2) becomes

$$u + x\frac{du}{dx} = F(u) \Rightarrow \frac{dx}{x} = \frac{du}{F(u) - u}$$ is separable.

Ex5 $2xyy' - y^2 + x^2 = 0$

Solution: Let $y = ux$, the equation becomes

$$2xux(u + u'x) - u^2x^2 + x^2 = 0 \Rightarrow 2u(u + xu') - u^2 + 1 = 0 \Rightarrow 2xuu' + u^2 + 1 = 0$$

$$\frac{2udu}{1 + u^2} = -\frac{dx}{x} \Rightarrow \ln(1 + u^2) = -\ln x + C' \Rightarrow 1 + u^2 = C/x$$

$$1 + \left(\frac{y}{x}\right)^2 = \frac{C}{x} \Rightarrow x^2 + y^2 = Cx$$

Ex6 $(x + y\cos\frac{y}{x})dx - x\cos\frac{y}{x}dy = 0$

Solution: Let $y = ux \Rightarrow (x + ux\cos u)dx - x\cos u(udx + xdu) = 0 \Rightarrow \cos u du = \frac{dx}{x}$

$$\sin u = \ln x + C \Rightarrow \sin\frac{y}{x} - \ln x = C$$

2. $M(x, y)$ and $N(x, y)$ are linear in x and y

Form: $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$ (2.3)

(1) If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, let $x = X + \alpha$, $y = Y + \beta$, equation (2.3) becomes

$$[(a_1X + b_1Y) + (a_1\alpha + b_1\beta + c_1)]dX + [(a_2X + b_2Y) + (a_2\alpha + b_2\beta + c_2)]dY = 0. \quad (2.4)$$

We choose $\begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0 \end{cases}$

Then equation (2.4) reduces to

$$(a_1X + b_1Y)\,dX + (a_2X + b_2Y)\,dY = 0$$

is a homogeneous equation.

(2) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k \neq \frac{c_1}{c_2}$, equation (2.3) becomes

$$[k(a_2x + b_2y) + c_1]\,dx + (a_2x + b_2y + c_2)\,dy = 0. \quad (2.5)$$

Let $v = a_2x + b_2y \Rightarrow dy = \frac{dv - a_2dx}{b_2}$, equation (2.5) becomes

$$(kv + c_1)dx + (v + c_2)\frac{dv - a_2dx}{b_2} = 0$$

$[kv + c_1 - \frac{a_2}{b_2}(v + c_2)]dx + \frac{v + c_2}{b_2}dv = 0$ is a separable one.

(3) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, equation (2.3) becomes

$$k(a_2x + b_2y + c_2)dx + (a_2x + b_2y + c_2)dy = 0. \quad (2.6)$$

a. If $a_2x + b_2y + c_2 = 0$, we get only a trivial solution.

b. If $a_2x + b_2y + c_2 \neq 0$, equation (2.6) reduces to $kdx + dy = 0 \Rightarrow y = -kx + C$.

Ex7 $(4x+3y+1)dx+(x+y+1)dy=0$

Solution: $\begin{cases} 4x+3y+1=0 \\ x+y+1=0 \end{cases} \Rightarrow x=2,\ y=-3$

Let $X=x-2$, $Y=y+3$, we have

$(4X+3Y)dX+(X+Y)dY=0$

Let $Y=uX \Rightarrow (4X+3uX)dX+(X+uX)(udX+Xdu)=0$

$$(4+4u+u^2)dX+X(1+u)du=0 \Rightarrow \frac{dX}{X}+\frac{1+u}{(u+2)^2}du=0$$

$$\frac{dX}{X}+\left[\frac{1}{u+2}-\frac{1}{(u+2)^2}\right]du=0 \Rightarrow \ln X+\ln(u+2)+\frac{1}{u+2}=C$$

$$\ln[X(u+2)]+\frac{1}{u+2}=C \Rightarrow \ln(2x+y-1)+\frac{x-2}{2x+y-1}=C$$

Ex8 $(2x-4y+5)y'+x-2y+3=0$

Solution : Let $u=x-2y \Rightarrow u'=1-2y' \Rightarrow y'=\frac{1-u'}{2}$

The equation becomes $(2u+5)\frac{1-u'}{2}+u+3=0$

$$(2u+5)(1-u')+2u+6=0 \Rightarrow 4u+11-(2u+5)u'=0$$

$$(4u+11)dx-(2u+5)du=0 \Rightarrow dx-\frac{2u+5}{4u+11}du=0$$

$$dx-\left(\frac{1}{2}-\frac{1/2}{4u+11}\right)du=0 \Rightarrow \int dx-\int\left(\frac{1}{2}-\frac{1/2}{4u+11}\right)du=C_1$$

$$x-\frac{u}{2}+\frac{1}{8}\ln(4u+11)=C_1 \Rightarrow 8x-4u+\ln(4u+11)=8C_1$$

$$8x-4(x-2y)+\ln[4(x-2y)+11]=8C_1 \Rightarrow 4x+8y+\ln(4x-8y+11)=C$$

Ex9 $(x-y-1)dx+(2x-2y-2)dy=0$

Solution: Dividing by $x-y-1 \Rightarrow dx+2dy=0 \Rightarrow x+2y=C$

3. Type of $y' = f(ax + by + c)$ (2.7)

Let $u = ax + by + c \Rightarrow u' = a + by' \Rightarrow y' = \dfrac{\frac{du}{dx} - a}{b}$, equation (2.7) becomes

$$\frac{\frac{du}{dx} - a}{b} = f(u) \Rightarrow \frac{du}{dx} - a = bf(u) \Rightarrow \frac{du}{dx} = a + bf(u) \Rightarrow \frac{du}{a + bf(u)} = dx \quad \text{is separable.}$$

Ex10 $y' = (x + y - 7)^2$ (2.8)

Solution: Let $u = x + y - 7 \Rightarrow u' = 1 + y'$, equation (2.8) has the result

$$u' - 1 = u^2 \Rightarrow \frac{du}{1 + u^2} = dx \Rightarrow \tan^{-1} u = x + C \Rightarrow x + y - 7 = \tan(x + C)$$

[Exercises] 1. $(2x + y - 2)y' = 4$

2. $y' = \sin^2(x - y)$

[Answers] 1. $y = 2\ln(2x + y) + C$ 2. $\tan(x - y) = x + C$

4. Isobaric equations

Definition: If $f(tx, t^m y, t^{m-1} y') = t^r f(x, y, y')$, we say that $f(x, y, y')$ is an isobaric function of weight r.

In particular, if $t = 1/x$, there follows

$$f(1, \frac{y}{x^m}, \frac{y'}{x^{m-1}}) = \frac{1}{x^r} f(x, y, y') \text{ or}$$

$$f(x, y, y') = x^r f(1, \frac{y}{x^m}, \frac{y'}{x^{m-1}}) \equiv x^r F(\frac{y}{x^m}, \frac{y'}{x^{m-1}}). \qquad (2.9)$$

A differential equation can be put into the form (2.9), then $F(\frac{y}{x^m}, \frac{y'}{x^{m-1}}) = 0$.

Solving this for $\dfrac{y'}{x^{m-1}}$, and multiplying by x^{m-1}, we have

$$y' = x^{m-1}\Phi\left(\frac{y}{x^m}\right) \tag{2.10}$$

Let $u = \dfrac{y}{x^m} \Rightarrow y = ux^m \Rightarrow \dfrac{dy}{dx} = x^m \dfrac{du}{dx} + mux^{m-1}$, (2.10) is reducible to

$$x^m \frac{du}{dx} + mux^{m-1} = x^{m-1}\Phi(u) \Rightarrow \frac{du}{\Phi(u) - mu} = \frac{dx}{x} \text{ is separable.}$$

[Note] If the weights m and $m-1$ are assigned to y and y' respectively, the term $x^a y^b y'^c$ has the weight $a + bm + c(m-1)$. If an expression is to be isobaric, all its terms must be isobaric and of the same weight.

Ex11 $2x^3 y' = 1 + \sqrt{1 + 4x^2 y}$.

Solution: The weights of each term are $3 + (m-1)$, 0, $\frac{1}{2}(0, 2+m)$, if $m = -2$, every term has the same weight. Let $y = ux^{-2} \Rightarrow y' = u'x^{-2} - 2ux^{-3}$, the equation becomes

$$2x^3(x^{-2}u' - 2ux^{-3}) = 1 + \sqrt{1+4u} \Rightarrow 2xu' - 4u = 1 + \sqrt{1+4u}$$

$$\frac{du}{1 + 4u + \sqrt{1+4u}} = \frac{dx}{2x} \Rightarrow \frac{du}{\sqrt{1+4u}\left(\sqrt{1+4u} + 1\right)} = \frac{dx}{2x} \Rightarrow \frac{d\left(\sqrt{1+4u} + 1\right)}{2\left(\sqrt{1+4u} + 1\right)} = \frac{dx}{2x}$$

$$\ln\left(\sqrt{1+4u} + 1\right) = \ln x + C' \Rightarrow \frac{\sqrt{1+4x^2 y} + 1}{x} = C$$

Ex12 $y^2 + (1 + xy)y' = 0$.

Solution: weight $2m, m-1, 1+m+m-1$, so $2m = m-1 \Rightarrow m = -1$.

Let $y = ux^{-1} \Rightarrow y' = u'x^{-1} - ux^{-2}$

$$x^{-2}u^2 + (1+u)(x^{-1}u' - x^{-2}u) = 0 \Rightarrow x^{-2}u^2 + x^{-1}u' - x^{-2}u + x^{-1}uu' - x^{-2}u^2 = 0$$

$$(1+u)u' = x^{-1}u \Rightarrow \frac{(1+u)du}{u} = \frac{dx}{x} \Rightarrow \left(\frac{1}{u} + 1\right)du = \frac{dx}{x} \Rightarrow \ln u + u = \ln x + C$$

$$\ln\frac{u}{x} + u = C \Rightarrow \ln y + xy = C$$

Ex13 $x^3y' - x^2y + y^2 = 0.$

Solution: weight $3+m-1,\ 2+m,\ 2m$, so $m+2=2m \Rightarrow m=2$, Let $y=ux^2, \Rightarrow y'=x^2u'+2xu$

$$x^3(x^2u'+2xu)-x^2(ux^2)+(ux^2)^2=0 \Rightarrow xu'+2u-u+u^2=0$$

$$\frac{du}{u+u^2}+\frac{dx}{x}=0 \Rightarrow \left(\frac{1}{u}-\frac{1}{u+1}\right)du+\frac{dx}{x}=0 \Rightarrow \ln\frac{ux}{u+1}=C'$$

$$\frac{y/x}{y/x^2+1}=C \Rightarrow \frac{xy}{y+x^2}=C$$

[Exercises] 1. $2x^2y' - x^2y^2 + 2xy + 1 = 0$

2. $x^3y' + 4x^2y + 1 = 0$

3. $(x+2x^2y)y' + 2y + 3xy^2 = 0$

4. $2xyy' = y^2 + \sqrt{y^4 - 4x^2}$

[Answers] 1. $x + x^2y = C(1-xy)$ 2. $x^2 + 2x^4y = C$ 3. $x^2y + x^3y^2 = C$ 4. $Cy^2 - x^2 = C^2$

III. Exact differential equations

1. Exact differential equations

A first order differential equations $M(x,y)dx + N(x,y)dy = 0$ is called exact if its left side is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, its solution is $u = C$

[Condition] $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Since $M = \frac{\partial u}{\partial x}, N = \frac{\partial u}{\partial y}$, by the assumption of continuity $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution can be obtained from the following :

$$u = \int_x M dx + k(y)$$

and from $\frac{\partial u}{\partial y} = N$, we can determine $k(y)$.

Ex14 $(x^3+3xy^2)\,dx+(3x^2y+y^3)\,dy=0$.

Solution: Since $\dfrac{\partial\,(x^3+3xy^2)}{\partial\,y}=\dfrac{\partial\,(3x^2y+y^3)}{\partial\,x}=6xy$, it is exact, and

$$u=\int_x (x^3+3xy^2)\,dx+k(y)=x^4/4+3x^2y^2/2+k(y)$$

$$\frac{\partial u}{\partial y}=N\Rightarrow 3x^2y+k'(y)=3x^2y+y^3,\Rightarrow k(y)=y^4/4$$

the solution is $x^4/4+3x^2y^2/2+y^4/4=C$

Ex15 $(\sin x\cosh y)\,dx-(\cos x\sinh y)\,dy=0,\ y(0)=0$.

Solution: Since $\dfrac{\partial\,(\sin x\cosh y)}{\partial\,y}=\dfrac{\partial\,(-\cos x\sinh y)}{\partial\,x}=\sin x\sinh y$, it is exact, and

$$u=\int_x \sin x\cosh y\,dx+k(y)=-\cos x\cosh y+k(y)$$

$$\frac{\partial u}{\partial y}=N\Rightarrow -\cos x\sinh y+k'(y)=-\cos x\sinh y\Rightarrow k(y)=C\Rightarrow u=-\cos x\cosh y=C$$

$y(0)=0\Rightarrow -\cos 0\cosh 0=C\Rightarrow C=-1$, The solution is $\cos x\cosh y=1$

[Exercises] 1. $\dfrac{dx}{\sqrt{x^2+y^2}}+\left(\dfrac{1}{y}-\dfrac{x}{y\sqrt{x^2+y^2}}\right)dy=0$

2. $\dfrac{(x+y)dx-(x-y)dy}{x^2+y^2}=0$

3. $\dfrac{dx}{\sqrt{xy}}+\left(\dfrac{2}{y}-\sqrt{\dfrac{x}{y^3}}\right)dy=0$

[Answers] 1. $y^2+2Cx=C^2$ 2. $\log\sqrt{x^2+y^2}+\tan^{-1}\dfrac{x}{y}=C$ 3. $\sqrt{\dfrac{x}{y}}+\log y=C$

2. Integrating factors

If $M(x, y)\,dx+N(x, y)\,dy=0$ is not exact, we multiply by $\mu(x, y)$, $\mu M(x, y)\,dx+\mu N(x, y)dy=0$ is exact. The function $\mu(x, y)$ is called an integrating factor.

[Case 1]If $\mu(x, y) = \mu(x)$, from the condition of exactness, we get

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \Rightarrow \mu \frac{\partial M}{\partial y} = N\mu' + \mu \frac{\partial N}{\partial x} \Rightarrow N\mu' = \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\frac{d\mu}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx$$

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ only depends on x, we have

$$\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$$

[Case 2] If $\mu(x, y) = \mu(y)$, from the condition of exactness, we get

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \Rightarrow \mu \frac{\partial M}{\partial y} + M\mu' = \mu \frac{\partial N}{\partial x} \Rightarrow \frac{d\mu}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} dy$$

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M}$ depends only on y, we have

$$\mu(y) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} dy}$$

Ex16 $ydx - xdy = 0$ (2.11)

Solution : $M = y,\ N = -x \Rightarrow \dfrac{\partial M}{\partial y} = 1,\ \dfrac{\partial N}{\partial x} = -1$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-(-1)}{-x} = -\frac{2}{x}$$

$$\mu = e^{\int -\frac{2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

$$u = \int_x \mu M dx + f(y) = \int_x x^{-2} y dx + f(y)$$

$$= -x^{-1} y + f(y)$$

$$\frac{\partial u}{\partial y} = \mu N \Rightarrow -x^{-1} + f'(y) = x^{-2} \cdot (-x) \Rightarrow f'(y) = 0 \Rightarrow f(y) = 0$$

The solution is $-x^{-1} y = C_1 \Rightarrow \dfrac{y}{x} = C$

Ex17 $2xydx+(4y+3x^2)\,dy=0, \quad y(0.2)=-1.5$

Solution: Since $\dfrac{\partial M/\partial y-\partial N/\partial x}{-M}=\dfrac{2x-6x}{-2xy}=\dfrac{2}{y}$ depends only on y, integrating factor is

$e^{\int_y \frac{2}{y}dy}=y^2$ and $u=\int_x 2xy^3dx+k(y)=x^2y^3+k(y)$

$\dfrac{\partial u}{\partial y}=\mu N \Rightarrow 3x^2y^2+k'(y)=y^2(4y+3x^2)\Rightarrow k(y)=y^4$, and the general solution is

$x^2y^3+y^4=C$ Substituting $y(0.2)=-1.5$, we have

$(0.2)^2(-1.5)^3+(-1.5)^4=C\Rightarrow C=4.9275\Rightarrow$ The solution is $x^2y^3+y^4=4.9275$

[Case 3] Form: $x^ry^s(mydx+nxdy)=0$ (2.12)

Since $d(x^ay^b)=x^{a-1}y^{b-1}(aydx+bxdy)$, (2.12) has an integrating factor $x^\alpha y^\beta$, where $\alpha+r=m-1$, $\beta+s=n-1$.

More generally, suppose $x^\alpha y^\beta$ is the integrating factor of the equation

$$x^ry^s(mydx+nxdy)+x^py^q(uydx+vxdy)=0. \quad (2.13)$$

Multiplying (2.13) by $x^\alpha y^\beta$ and rearranging the terms, we have

$$(mx^{\alpha+r}y^{\beta+s+1}+ux^{\alpha+p}y^{\beta+q+1})\,dx+(nx^{\alpha+r+1}y^{\beta+s}+vx^{\alpha+p+1}y^{\beta+q})\,dy=0.$$

This is exact, provided $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, i.e.,

$$m(\beta+s+1)x^{\alpha+r}y^{\beta+s}+u(\beta+q+1)x^{\alpha+p}y^{\beta+q}=n(\alpha+r+1)x^{\alpha+r}y^{\beta+s}+v(\alpha+p+1)x^{\alpha+p}y^{\beta+q}$$

for all values of x and y, we have

$$m(\beta+s+1)=n(\alpha+r+1), \quad u(\beta+q+1)=v(\alpha+p+1).$$

The two equations can be solved for α and β.

Ex18 $x^4y(3ydx+2xdy)+x^2(4ydx+3xdy)=0$

Solution:

$(3x^4y^2+4x^2y)dx+(2x^5y+3x^3)dy=0$

Multiplying $x^{\alpha}y^{\beta}$

$(3x^{\alpha+4}y^{\beta+2}+4x^{\alpha+2}y^{\beta+1})dx+(2x^{\alpha+5}y^{\beta+1}+3x^{\alpha+3}y^{\beta})dy=0$

$M=3x^{\alpha+4}y^{\beta+2}+4x^{\alpha+2}y^{\beta+1},\ N=2x^{\alpha+5}y^{\beta+1}+3x^{\alpha+3}y^{\beta}$

To be exact $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

$3(\beta+2)x^{\alpha+4}y^{\beta+1}+4(\beta+1)x^{\alpha+2}y^{\beta}=2(\alpha+5)x^{\alpha+4}y^{\beta+1}+3(\alpha+3)x^{\alpha+2}y^{\beta}$

$3(\beta+2)=2(\alpha+5),\quad 4(\beta+1)=3(\alpha+3)\Rightarrow \alpha=1,\beta=2$

$u=\int_x Mdx+f(y)=\int_x(3x^5y^4+4x^3y^3)dx+f(y)=\dfrac{1}{2}x^6y^4+x^4y^3+f(y)$

$\dfrac{\partial u}{\partial y}=N\Rightarrow 2x^6y^3+3x^4y^2+f'(y)=2x^6y^3+3x^4y^2\Rightarrow f'(y)=0\Rightarrow f(y)=0$

The solution is $\dfrac{1}{2}x^6y^4+x^4y^3=C$

Ex19 $2ydx+xdy+xy(3ydx+2xdy)=0$

Solution: Multiplying $x^{\alpha}y^{\beta}$ and rearranging the terms, we get

$(2x^{\alpha}y^{\beta+1}+3x^{\alpha+1}y^{\beta+2})dx+(x^{\alpha+1}y^{\beta}+2x^{\alpha+2}y^{\beta+1})dy=0$

For exactness $2(\beta+1)x^{\alpha}y^{\beta}+3(\beta+2)x^{\alpha+1}y^{\beta+1}=(\alpha+1)x^{\alpha}y^{\beta}+2(\alpha+2)x^{\alpha+1}y^{\beta+1}$

$2(\beta+1)=\alpha+1,\quad 3(\beta+2)=2(\alpha+2)\Rightarrow \alpha=1,\beta=0$

$u=\int_x(2xy+3x^2y^2)dx+k(y)=x^2y+x^3y^2+k(y)$

$\dfrac{\partial u}{\partial y}=\mu N\Rightarrow x^2+2x^3y+k'(y)=x^2+2x^3y\Rightarrow k(y)=C$

The general solution is $x^2y+x^3y^2=C$

IV. Linear differential equations of first order

1. Form: $y' + p(x)y = r(x)$ (2.14)

If $r(x) = 0$, (2.14) is said to be homogeneous, otherwise it is said to be nonhomogeneous. (2.14) is equivalent to

$$\frac{d}{dx}[F(x)y] = F(x)r \tag{2.15}$$

$$Fy' + F'y = Fr \Rightarrow y' + \frac{F'}{F}y = r$$

Comparing with (2.14), we have

$$\frac{F'}{F} = p \Rightarrow \frac{dF}{F} = pdx \Rightarrow F = e^{\int pdx}$$

Integrating (2.15), we get the solution:

$$Fy = \int Frdx + C \Rightarrow y = F^{-1}\int Frdx + CF^{-1}$$

$$y = e^{-\int pdx}\left[\int e^{\int pdx} rdx + C\right] \quad \text{or} \quad y = e^{-h}[\int e^{h} rdx + C], \text{ where } h = \int pdx$$

Ex20 $y' - y = e^{2x}$

Solution: $h = \int -dx = -x$

$$y = e^{x}[\int e^{-x}e^{2x}dx + C] = e^{2x} + Ce^{x}$$

Ex21 $y' + 2y = e^{x}(3\sin 2x + 2\cos 2x)$

Solution: $h = \int 2dx = 2x$

$$y = e^{-2x}[\int e^{2x}e^{x}(3\sin 2x + 2\cos 2x)dx + C] = e^{x}\sin 2x + Ce^{-2x}$$

Ex22 $y' + y\tan x = \sin 2x,\ y(0) = 1$

Solution: $h = \int \tan x dx = \ln\sec x$

$$y = e^{-\ln\sec x}[\int e^{\ln\sec x}\sin 2x dx + C] = \cos x\int 2\sin 2x dx + C\cos x = -2\cos^2 x + C\cos x$$

$$y(0) = 1 \Rightarrow 1 = -2\cos^2(0) + C\cos(0) \Rightarrow C = 3$$

$$y = -2\cos^2 x + 3\cos x$$

2. Reducible to linear form: Bernoulli equations

Form: $y' + p(x)y = g(x)y^a$ (a is any real number) (2.16)

$$(2.16) \Rightarrow y^{-a}y' + p(x)y^{1-a} = g(x) \quad (2.17)$$

Let $u = y^{1-a} \Rightarrow u' = (1-a)y^{-a}y'$, (2.17) becomes

$$\frac{1}{1-a}u' + pu = g(x) \Rightarrow u' + (1-a)pu = (1-a)g(x)$$

This is linear in u.

Ex23 $y' - Ay = -By^2$ (A, B are constants.)

Solution: $y^{-2}y' - Ay^{-1} = -B$

Let $u = y^{-1} \Rightarrow u' = -y^{-2}y'$, there follows

$$u' + Au = B \Rightarrow h = \int Adx = Ax \Rightarrow u = e^{-Ax}(\int e^{Ax}Bdx + C) = \frac{B}{A} + Ce^{-Ax}$$

$$y = \frac{1}{B/A + Ce^{-Ax}}$$

Ex24 $(e^y + x)y' = 1$

Solution: The equation can be written in the form

$$\frac{dx}{dy} - x = e^y \text{ which is linear in } x.$$

$$h = \int -dy = -y, \Rightarrow x = e^y[\int e^{-y}e^y dy + C] = e^y(y + C)$$