Laplace Transform

Chapter 1 Introduction to Laplace Transform

I. Laplace Transform of Basic Functions

Definition: The Laplace transform of a function $f(t)$ is defined as

$$F(s)=\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt$$

Original function	Transformed function
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$

Original function	Transformed function
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

1. $\mathcal{L}[1]=\int_0^\infty e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=\frac{1}{s}$

2. $\mathcal{L}[t^a]=\int_0^\infty t^a e^{-st}dt=\int_0^\infty (\frac{u}{s})^a e^{-u}\frac{du}{s}=\frac{1}{s^{a+1}}\int_0^\infty u^a e^{-u}du=\frac{\Gamma(a+1)}{s^{a+1}}$

3. $\mathcal{L}[e^{at}]=\int_0^\infty e^{at}e^{-st}dt=\frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty=\frac{1}{s-a}$

4. $\mathcal{L}[e^{iat}]=\frac{1}{s-ia}\Rightarrow\mathcal{L}[\cos at+i\sin at]=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}$

   $\therefore\mathcal{L}[\cos at]=\frac{s}{s^2+a^2}$, and $\mathcal{L}[\sin at]=\frac{a}{s^2+a^2}$

5. $\mathcal{L}[\sinh at]=\mathcal{L}[\frac{e^{at}-e^{-at}}{2}]=\frac{1}{2}(\frac{1}{s-a}-\frac{1}{s+a})=\frac{a}{s^2-a^2}$

   $\mathcal{L}[\cosh at]=\mathcal{L}[\frac{e^{at}+e^{-at}}{2}]=\frac{1}{2}(\frac{1}{s-a}+\frac{1}{s+a})=\frac{s}{s^2-a^2}$

[Note] $\Gamma(x)=\int_0^{\infty} e^{-u}u^{x-1}du$ is called Gamma function.

The properties of Gamma function

$\Gamma(a+1)=a\Gamma(a)$.

$\Gamma(1)=1$.

$\Gamma(n+1)=n!$, n is a natural number.

$\Gamma(\frac{1}{2})=\sqrt{\pi}$.

$$\Gamma(a+1)=\int_0^{\infty} e^{-u}u^{(a+1)-1}du=\int_0^{\infty} e^{-u}u^{a}du=-(e^{-u}u^{a}\Big|_0^{\infty}-a\int_0^{\infty} e^{-u}u^{a-1}du)=a\int_0^{\infty} e^{-u}u^{a-1}du=a\Gamma(a)$$

$$\Gamma(1)=\int_0^{\infty} e^{-u}u^{1-1}du=\int_0^{\infty} e^{-u}du=-e^{-u}\Big|_0^{\infty}=1$$

When n is a natural number, then

$$\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=n(n-1)(n-2)\Gamma(n-2)$$
$$=\cdots=n(n-1)(n-2)\cdots\times 2\times 1\times\Gamma(1)=n!$$

$$\Gamma(\frac{1}{2})=\int_0^{\infty} e^{-u}u^{\frac{1}{2}-1}du=\int_0^{\infty} e^{-u}u^{-\frac{1}{2}}du=2\int_0^{\infty} e^{-u}d\sqrt{u}=2\int_0^{\infty} e^{-x^2}dx=2\sqrt{\int_0^{\infty} e^{-x^2}dx\cdot\int_0^{\infty} e^{-y^2}dy}$$

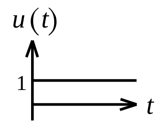
$$=2\sqrt{\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy}=2\sqrt{\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta}=2\sqrt{\frac{1}{2}\int_0^{\frac{\pi}{2}}(-e^{-r^2})_0^{\infty}d\theta}=2\sqrt{\frac{1}{2}\int_0^{\frac{\pi}{2}}1d\theta}$$

$$=2\sqrt{\frac{1}{2}\cdot\frac{\pi}{2}}=\sqrt{\pi}$$

III. The Laplace Transform of Special Functions

1. Unit step function (Heaviside function)

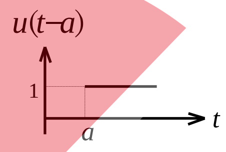
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}\,[u(t)] = \int_0^\infty u(t)e^{-st}dt$$

$$= \int_0^\infty e^{-st}dt = \frac{1}{s}$$

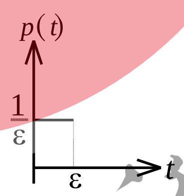
$$\Rightarrow u(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$



$$\mathcal{L}\,[u(t-a)] = \frac{e^{-as}}{s}$$

2. Unit impulse function (Dirac delta function)

(1) square wave function $p(t) = \begin{cases} \frac{1}{\varepsilon} & 0 \le t \le \varepsilon \\ 0 & t > \varepsilon \end{cases}$



$$\mathcal{L}\,[p(t)] = \int_0^\infty p(t)e^{-st}dt = \int_0^\infty \frac{1}{\varepsilon}[u(t) - u(t-\varepsilon)]e^{-st}dt$$

$$= \frac{1}{\varepsilon}\left(\frac{1}{s} - \frac{e^{-\varepsilon s}}{s}\right) = \frac{1-e^{-\varepsilon s}}{s\varepsilon}$$

(2) unit impulse function $\delta(t) = \lim_{\varepsilon \to 0} p(t)$ (also called singular function, Dirac delta function)

$$\mathcal{L}\,[\delta(t)] = \mathcal{L}\,[\lim_{\varepsilon\to 0} p(t)] = \lim_{\varepsilon\to 0}\{\mathcal{L}\,[p(t)]\} = \lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{s\varepsilon} = \lim_{\varepsilon\to 0}\frac{se^{-\varepsilon s}}{s} = 1$$

Properties:

(i) $\int_0^\infty \delta(t)dt = u(t)$

(ii) $\int_0^\infty g(t)\delta(t-a)dt = g(a)$

(iii) $\int_0^\infty g(t)\delta(t)dt = g(0)$

$\mathcal{L}\,[\delta(t-a)] = e^{-as}$