9.4 Power series (冪級數)

Define:

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$ is called a power series centered at $c \in \mathbb{R}$.

If $c = 0$, then $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ is called a power series.

Theorem: Convergence of a power series

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, (or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$), then

(1) $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if $|x-c| < \frac{1}{L} \equiv R$.

(2) $\sum_{n=0}^{\infty} a_n(x-c)^n$ diverges if $|x-c| > \frac{1}{L}$.

(3) Each endpoint $x = c \pm R$ must be tested separately for convergence or divergence.

Define:

(1) The number $R = \frac{1}{L}$ is the radius of convergence (收斂半徑) of the power series.

(2) The set of all values of x for which the power series converges is the interval of convergence (收斂區間) of the power series.

Ex 1: Find the interval of convergence of $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$.

Theorem:

If $s(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$ converges for $|x-c|<R$, then $\forall x\in(c-R,c+R)$

(1) $s'(x)=\sum_{n=0}^{\infty}[a_n(x-c)^n]'=\sum_{n=1}^{\infty}na_n(x-c)^{n-1}$

(2) $\int_c^x s(t)dt=\sum_{n=0}^{\infty}\int_c^x a_n(t-c)^n dt=\sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-c)^{n+1}$.

Ex 2: Find the power series and the interval of convergence of $\ln(1+x)$.

Ex 3: Find (1) $\sum_{n=1}^{\infty}nx^{n-1}$ (2) $\sum_{n=0}^{\infty}(n+1)^2 x^n$