7.2 Volumes of solids of revolution

1. Volumes of solids with known cross sections

Theorem: A solid S with cross sectional area $A(x)$ at each point perpendicular to the x-axis on $[a,b]$ has volume

$$V=\int_a^b A(x)dx\,.$$

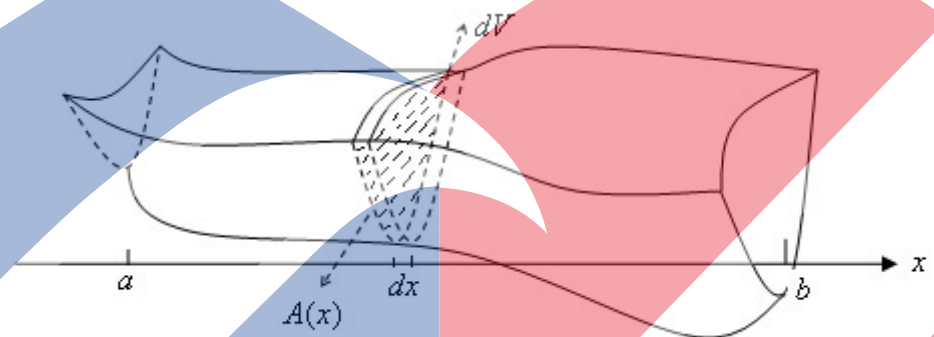


Figure 7.12

Ex 1: Show that the volume of a sphere of radius r is $V=\frac{4}{3}\pi r^3$.

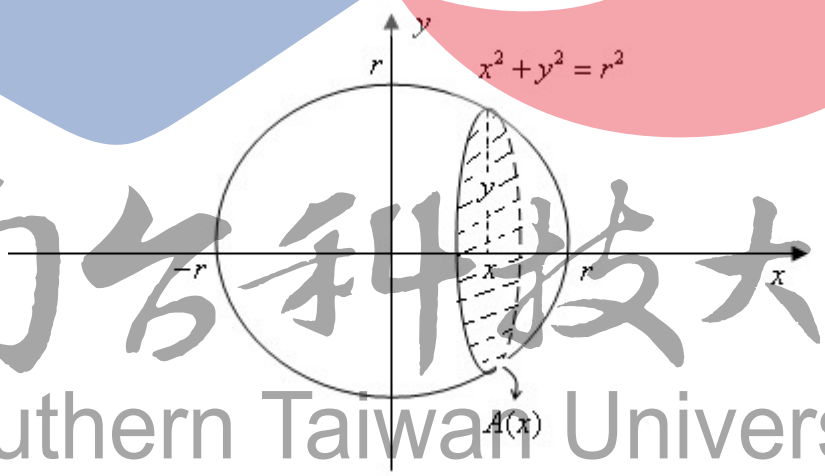


Figure 7.13

2. The disk method

Theorem: Suppose that $R=\{(x.y)\,|\,a\le x\le b, 0\le y\le f(x)\}$. Then the solid formed by revolving R about the x-axis has volume

$$V=\pi\int_a^b (f(x))^2\,dx.$$

$$\left[\!\left[\because V=\int_a^b A(x)dx=\int_a^b \pi(f(x))^2dx=\pi\int_a^b (\text{離轉軸距離})^2dx\right]\!\right]$$

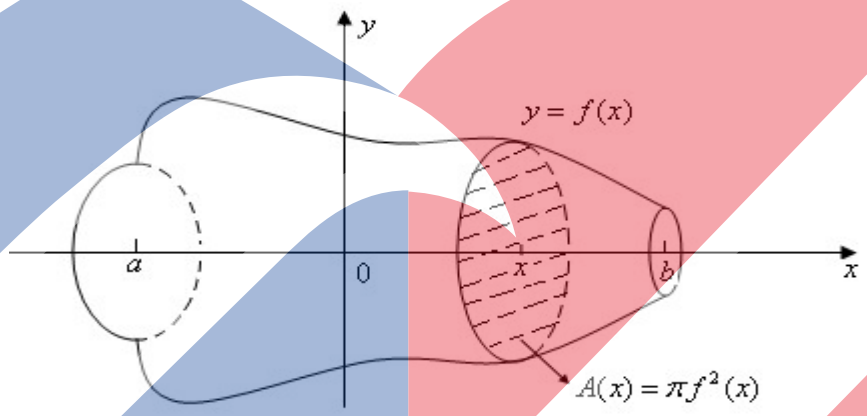


Figure 7.14

Ex 2: Find the volume of the solid formed by revolving the region bounded by the graphs of $y=\dfrac{\sqrt{3x}}{\sqrt{x^3+125}}$, $x=1$, $x=4$ and x-axis about the x-axis.

Corollary: Suppose that $R=\{(x,y)\,|\,a\le x\le b, 0\le g(x)\le y\le f(x)\}$, then the solid formed by revolving R about the x-axis has volume

$$V=\pi\int_a^b (f(x))^2-(g(x))^2\,dx$$

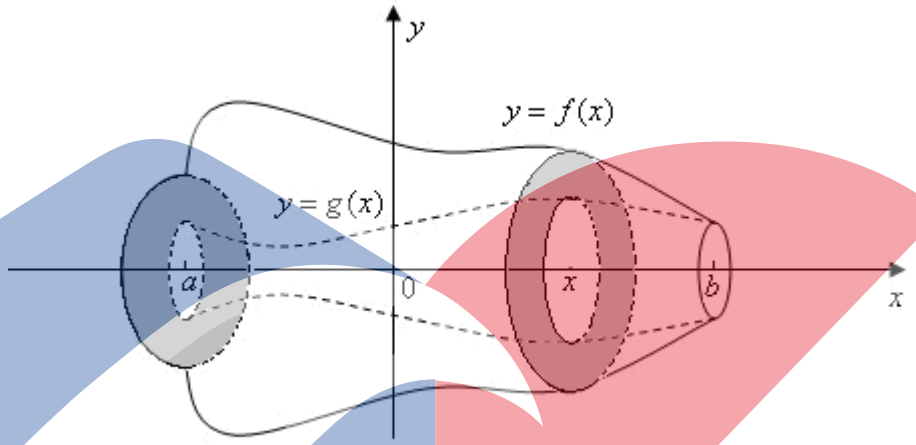


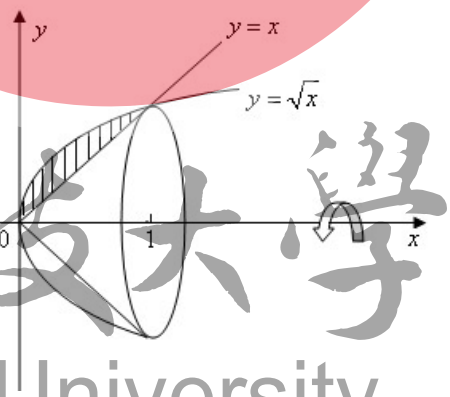
Figure 7.15

Ex 3: Let R be the region bounded by the graphs of $y=\sqrt{x}$ and $y=x$. Find the volume of the solid generated when R is revolved about the (a) x-axis (b) y-axis (c) line $y=-1$.

Sol:

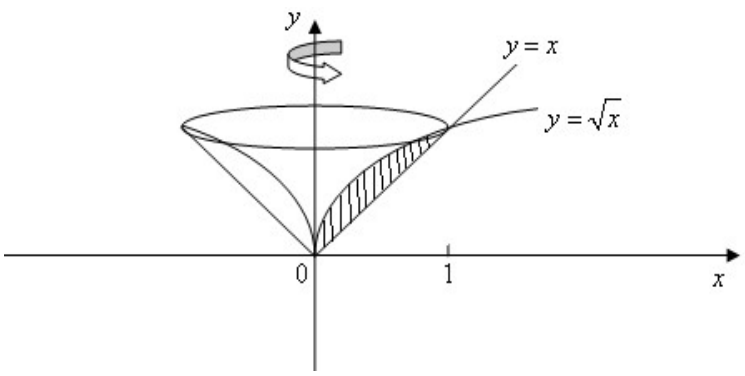
(a)

$$V=\pi\int_0^1 (\sqrt{x})^2-(x)^2\,dx$$



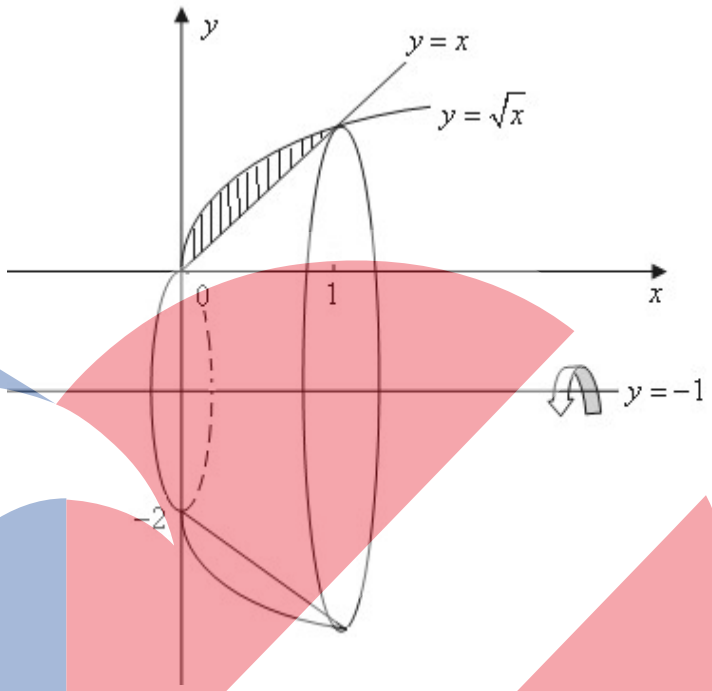
(b)

$$V=\pi\int_0^1 y^2-y^4\,dy$$



(c)

$$V=\pi\int_0^1(\sqrt{x}+1)^2-(x+1)^2dx$$



3. The shell method

Theorem: Suppose that $R=\{(x,y)\,|\,0\le a\le x\le b, 0\le y\le f(x)\}$. Then the solid generated by revolving R about the y-axis has volume

$$V=2\pi\int_a^b xf(x)dx\,.$$

$$\left[\because V=\int_a^b A(x)dx=\int_a^b(\text{圓周長})(\text{對應高})\text{dx}=2\pi\int_a^b xf(x)dx=2\pi\int_a^b(\text{離軸心長度})(\text{對應高度})\text{dx}\right]$$

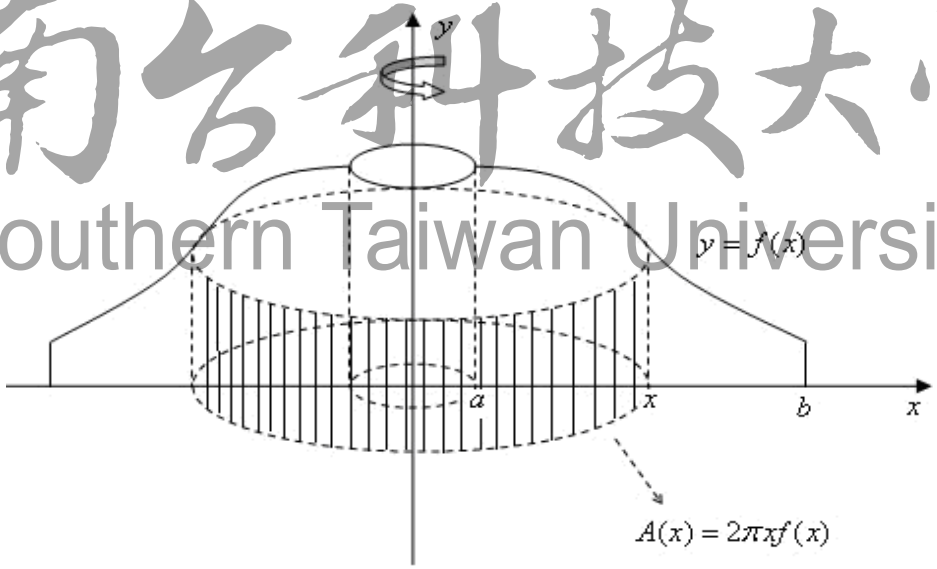


Figure 7.16

Ex 4: Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^2$ and x-axis $(0 \le x \le 1)$ about the y-axis.

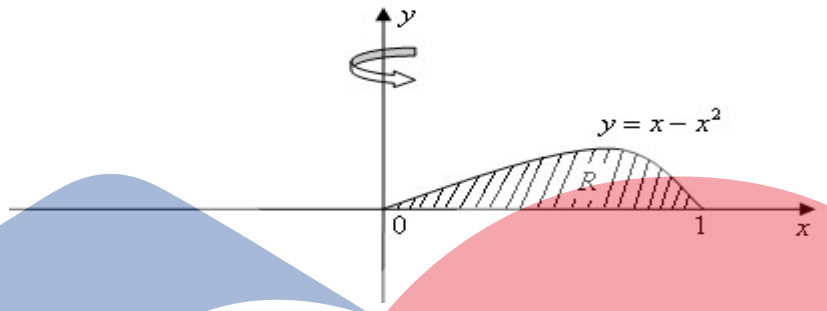


Figure 7.17

Corollary: Suppose that $R = \{(x, y) \mid 0 \le a \le x \le b, 0 \le g(x) \le y \le f(x)\}$, then the solid fromed by revolving R about the y-axis has volume

$$V = 2\pi \int_a^b x(f(x) - g(x))dx$$

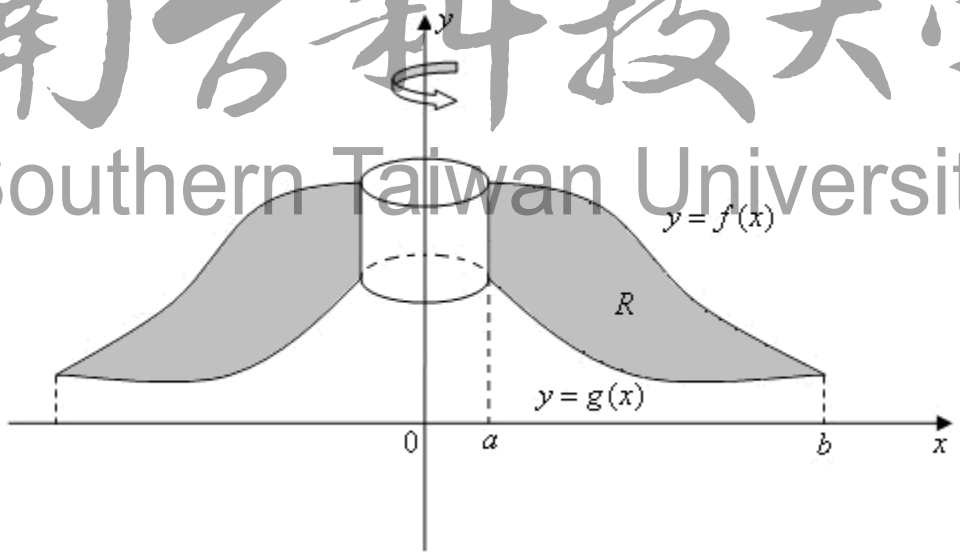


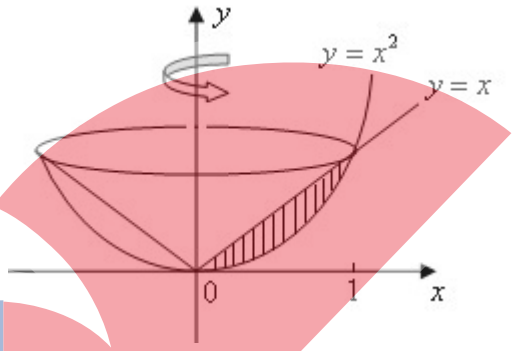
Figure 7.18

Ex 5: Find the volume of the solid formed by revolving the region bounded by the curves $y = x$ and $y = x^2$ about: (a) the y-axis (b) the line $x = -1$.

Sol:

(a)

$$V = 2\pi\int_0^1 x(x - x^2)dx$$



(b)

$$V = 2\pi\int_0^1 (x - (-1))(x - x^2)dx$$