11 Multiple integrals

11.1 Double Integrals (雙重積分)

Def: Let $f(x,y)$ be defined on a closed rectangle $\Omega=[a,b]\times[c,d]\subseteq\mathbb{R}^2$. We divide Ω into subrectangles $\Omega_{11},\Omega_{12},\cdots,\Omega_{1n},\Omega_{21},\cdots,\Omega_{mn}$, which $\Omega_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$. Let

$\Delta A_{ij}=\Delta x_i\cdot\Delta y_j$ be the area of Ω_{ij}. $\forall(s_i,t_j)\in\Omega_{ij}$, the double integral of f over Ω is

$$\iint_{\Omega} f(x,y)dA=\lim_{\|\Delta A_{ij}\|\to 0}\sum_{j=1}^{n}\sum_{i=1}^{m} f(s_i,t_j)\Delta x_i\Delta y_j$$

if the limit exists.

Def: Double integrals over general regions

Let $f(x,y)$ be defined on a bounded region Ω. Then $\exists\Omega^*=[a,b]\times[c,d],\ni\Omega\subseteq\Omega^*$. We define a new function

$$F(x,y)=\begin{cases} f(x,y), & if \quad (x,y)\in\Omega \\ 0, & if \quad (x,y)\in\Omega^*\setminus\Omega\end{cases}$$

If $\iint_{\Omega^*} F(x,y)dA$ exists, then $\iint_{\Omega^*} F(x,y)dA=\iint_{\Omega} f(x,y)dA$.

Concept: (1) If f is integrable over Ω and $f(x,y)\ge 0,\forall(x,y)\in\Omega$, ➔ the volume of the solid region that lies above Ω and below the graph of f is

$$V=\iint_{\Omega} f(x,y)dA$$

(2) If $f(x,y)=1$, ➔ $\iint_{\Omega} dA=|\Omega|$. (the area of Ω)

Ex 1: Find the volume of the solid region bounded above by the plane $z = 3$ and below by the circular region Ω given by $x^2 + y^2 \le 4$.

Properties of double integrals:

Let f and g be integrable over Ω.

(1) $\iint\limits_{\Omega} cf(x, y)dA = c\iint\limits_{\Omega} f(x, y)dA, \forall c \in \mathbb{R}$.

(2) $\iint\limits_{\Omega} f(x, y) \pm g(x, y)dA = \iint\limits_{\Omega} f(x, y)dA \pm \iint\limits_{\Omega} g(x, y)dA$.

(3) If $\Omega = \Omega_1 \cup \Omega_2$ and $|\Omega_1 \cap \Omega_2| = 0$, ➔ $\iint\limits_{\Omega} f(x, y)dA = \iint\limits_{\Omega_1} f(x, y)dA + \iint\limits_{\Omega_2} f(x, y)dA$.

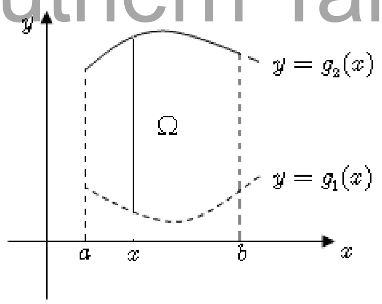
(4) $\iint\limits_{\Omega} f(x, y)dA \ge 0$, if $f(x, y) \ge 0, \forall (x, y) \in \Omega$.

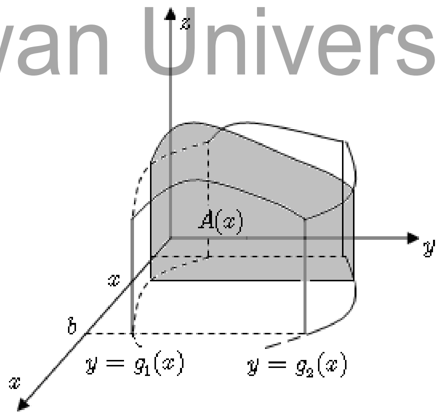
(5) $\iint\limits_{\Omega} f(x, y)dA \ge \iint\limits_{\Omega} g(x, y)dA$, if $f(x, y) \ge g(x, y), \forall (x, y) \in \Omega$.

Ex 2: Let $f(x, y) = \begin{cases} 1, 0 \le y < 1 \\ 2, 1 \le y \le 2 \end{cases}$, find $\iint\limits_{\Omega} f(x, y)dA$, where $\Omega = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 2\}$.

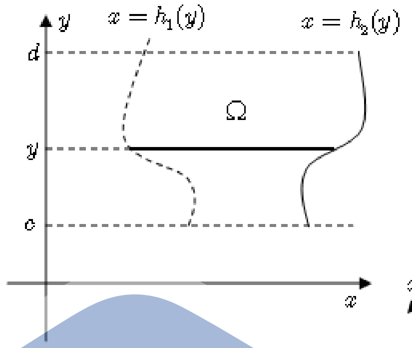
Evaluation of double integrals:

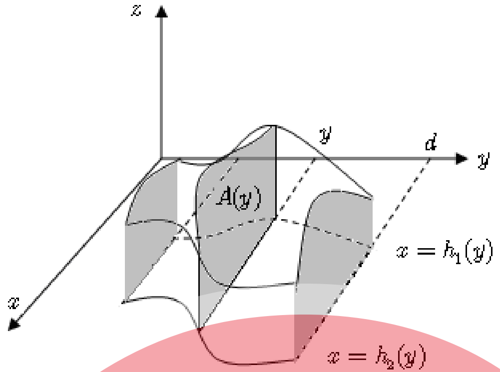
(1) If $\Omega = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$, ➔ $\iint\limits_{\Omega} f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$.





(2) If $\Omega = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$, ➔ $\iint\limits_{\Omega} f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$.



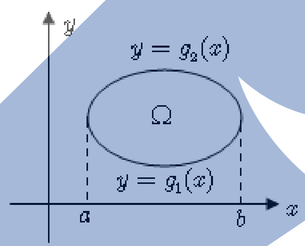


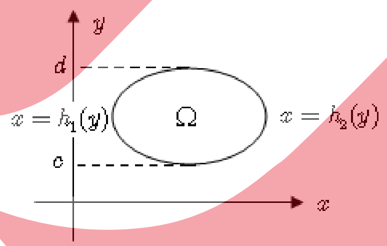
(3) Fubini's theorem:

Let f be continuous on Ω and

$\Omega = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\} = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

➔ $\iint\limits_{\Omega} f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$.





Ex 3: Evaluate $\iint\limits_{\Omega} 2xydA$, $\Omega = [0,1] \times [0,2]$.

Ex 4: Find the volume of the solid region R bounded by plane $x + 2y + z = 2$ and the three coordinate planes.

Ex 5: Evaluate $\int_0^1 \int_y^1 e^{-x^2} dxdy$