# 4.8 Newton's method

Claim: Find $r \in \mathbb{R}$, $\ni f(r) \approx 0$.

Algorithm: Newton's method

Input: $\varepsilon$ : tolerant(容許誤差),

Max: the maximum number of Newton iterations.

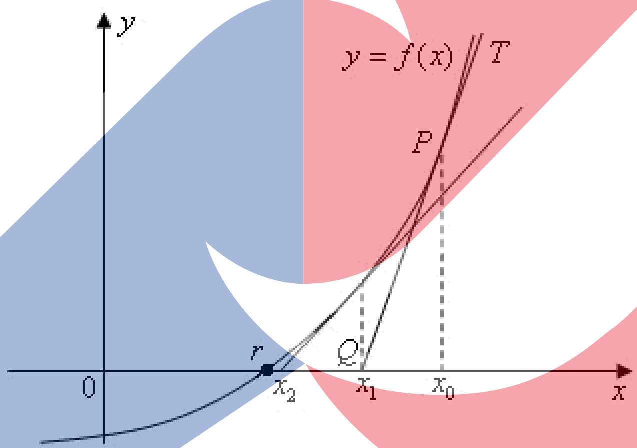
Step 1. Choose an initial estimate $x_0$ that is "close to " $r$.

Step 2. For $n = 0, 1, 2, \cdots, Max$ do

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Step 3. If $|f(x_{n+1})| < \varepsilon$, then $r \approx x_{n+1}$

Else go to Step 2.




Ex 1. Use Newton's method to solve $x^3 + 2x + 1 = 0$.

Sol：Let $f(x) = x^3 + 2x + 1$ ，then $f'(x) = 3x^2 + 2$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\,3} + 2x_n + 1}{3x_n^{\,2} + 2}$$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1}$ |
|---|---|---|---|---|
| 0 | 0.0 | 1.0 | 2.0 | −0.5 |
| 1 | −0.5 | −1.125 | 2.75 | −0.09091 |
| 2 | −0.09091 | 0.8174 | 2.0248 | −0.3128 |
| 3 | −0.3128 | 0.3438 | 2.2935 | −0.4627 |
| 4 | −0.4627 | −0.02447 | 2.6423 | −0.4534 |
| 5 | −0.4534 | $-1.085 \times 10^{-4}$ | | |

The solution is about $r \approx -0.4534$ 。

Ex 2. Use Newton method to approximate $\sqrt[5]{2}$, accurate to within $10^{-3}$.

Sol： Let $x=\sqrt[5]{2}$ ，then $x^5=2$ or $x^5-2=0$.

Assume that $f(x)=x^5-2$ ，then $f'(x)=5x^4$ 。

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1}$ |
|---|---|---|---|---|
| 0 | 1.0 | −1.0 | 5.0 | 1.2000 |
| 1 | 1.2000 | 0.4883 | 10.368 | 1.1529 |
| 2 | 1.1529 | 0.03685 | 8.8336 | 1.1487 |
| 3 | 1.1487 | $1.4320\times10^{-5}$ | | |

From above the table, we obtain $\sqrt[5]{2}\approx1.1487$ 。