2.4 Continuity

Def: $f(x)$ is continuous at a $\Leftrightarrow$ (1) $f(a)$ is defined.

(2) $\lim\limits_{x\to a} f(x)$ exists.

(3) $\lim\limits_{x\to a} f(x) = f(a)$

$$\Leftrightarrow \lim_{x\to a} f(x) = f(a)$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \ni \ |x-a| < \delta, \Rightarrow |f(x) - f(a)| < \varepsilon$$

Def: If f is not continuous at a ➔ f is discontinuous at a.

Ex 1: Where is each of the following functions discontinuous?

(1) $f(x) = \dfrac{1}{x-1}$

(2) $g(x) = \begin{cases} \frac{1}{x-1}, x \neq 1 \\ 0, \ x = 1 \end{cases}$

(3) $h(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ 0, \ x = 1 \end{cases}$

(4) $I(x) = \begin{cases} \frac{x^2-1}{x-1}, x \neq 1 \\ 2, \ x = 1 \end{cases}$

Def: If f is continuous on an interval I $\Leftrightarrow$ f is continuous at every point in I.

Theorem: If f and g are continuous at a, then (1) $cf, \forall c \in \mathbb{R}$ (2) $f \pm g$ (3) $f \bullet g$ (4) $\dfrac{f}{g}, g(a) \neq 0$ are also continuous at a.

Ex 2: Any polynomial is continuous function.

Ex 3: Discuss the continuity for (1) $f(x) = \begin{cases} x+2, x \neq 2 \\ 1 \quad , x = 2 \end{cases}$ (2) $g(x) = \begin{cases} \dfrac{1}{x}, \ x > 0 \\ -1, x \leq 0 \end{cases}$

(3) $h(x) = \dfrac{4x^3 - 3x^2 + 1}{x^2 - 3x + 2}$

Theorem: If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(L)$$

Ex 4: $\lim_{x \to 1}(x + \sqrt{2 - x^2})^3 = 8$

Ex 5: If $f(x) = \begin{cases} x, x \neq 0 \\ 1, x = 0 \end{cases}$, and $g(x) = x^2 - 1$, find $\lim_{x \to 1} f(g(x))$

Ex 6: Evaluate $\lim_{x \to 1} \sin^{-1}\left(\frac{1 - \sqrt{x}}{1 - x}\right)$

The intermediate value theorem(中間值定理):

(1) f is continuous on $[a,b]$

(2) $\forall k$ between $f(a)$ and $f(b)$,

then $\exists c \in [a,b]$, $\ni f(c) = k$.

Bolzano's theorem or Locating root theorem

(1) f is continuous on $[a,b]$

(2) $f(a) \cdot f(b) < 0$

➔ $\exists c \in (a,b)$, $\ni f(c) = 0$

Ex 7: Show that the function $f(x) = x^3 + x + 1$ has a zero in $(-1,1)$