Chapter 3 Vector Integral Calculus

I. Line integrals

1. Definition

A line integral of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C is

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$$

In terms of components

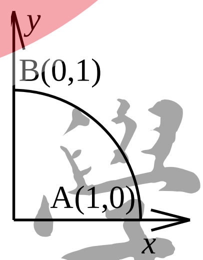
$$\int_C \mathbf{F}(\mathbf{r})\cdot d\,\mathbf{r} = \int_C F_1dx + F_2dy + F_3dz$$

If x, y, and z are function of t, we have

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\,\mathbf{r} = \int_C \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

Ex. 1.

Find the value of the line integral $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$, when $\mathbf{F} = -y\mathbf{i} + xy\mathbf{j}$ and C is the circular arc as shown from A to B.



Solution: We may represent C by

$$\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} \quad 0 \le t \le \pi/2$$

and

$$\mathbf{F} = -\sin t\mathbf{i} + \cos t\sin t\mathbf{j}$$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} (-\sin t\mathbf{i} + \cos t\sin t\mathbf{j})\cdot(-\sin t\mathbf{i} + \cos t\mathbf{j})dt$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 t + \cos^2 t\sin t)dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}dt - \int_0^{\frac{\pi}{2}} \cos^2 t d\cos t$$

$$= \frac{t}{2} - \frac{\sin 2t}{4} - \frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{3}$$

Ex. 2.

Find the value of the line integral $\int_C f ds$, when $f = x^2 + x^3 y$ and C is the line $y = 2x$ from (0, 0) to (2, 4).

Solution: We may represent C by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}dx = \sqrt{1 + 2^2}dx = \sqrt{5}dx$$

and

$$f = x^2 + x^3 y = x^2 + x^3(2x) = x^2 + 2x^4$$

$$\int_C f ds = \int_0^2 (x^2 + 2x^4)\sqrt{5}dx$$

$$= \sqrt{5}\left(\frac{x^3}{3} + \frac{2x^5}{5}\right)\Bigg|_0^2 = \sqrt{5}(\frac{8}{3} + \frac{64}{5}) = \frac{232\sqrt{5}}{15}$$

[Exercise] Evaluate the line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, with $\mathbf{F} = 5z\mathbf{i} + xy\mathbf{j} + x^2 z\mathbf{k}$ from point A(0, 0, 0) to point B(1, 1, 1) along

(1) C: the straight line $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$.

(2) C: the parabolic arc $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$.

[Solution](1) $\mathbf{F} = 5t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$

$$\int_C \mathrm{F}(\mathrm{r}) \cdot d\,\mathrm{r} = \int_0^1 (5t + t^2 + t^3)dt = 37/12$$

(2) $\mathbf{F} = 5t^2\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + 2t\mathbf{k})dt$

$$\int_C \mathrm{F}(\mathrm{r}) \cdot d\,\mathrm{r} = \int_0^1 (5t^2 + t^2 + 2t^5)dt = 7/3$$

2. Conservative fields

A vector field **F** is said to be conservative if there can be found some scalar ϕ such that $\mathbf{F} = \nabla\phi$, i.e., $F_1 = \dfrac{\partial\phi}{\partial x}$, $F_2 = \dfrac{\partial\phi}{\partial y}$, $F_3 = \dfrac{\partial\phi}{\partial z}$. Then ϕ is called a potential function or simply potential for **F**, and

$$\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

The line integral from A to B along a curve C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^B d\phi = \phi(B) - \phi(A)$$

This shows that the value of line integral is simply the difference of the value of ϕ at the two endpoints of C and is independent of the path C. Hence the line integral around a closed curve of a conservative field is zero.

Since

$$\nabla\times(\nabla\phi) = 0 \Rightarrow \nabla\times\mathbf{F} = 0$$

There follows a fact that if and only if $\nabla\times\mathbf{F} = 0$, the vector field **F** is conservative.

Ex. 3.

Find the line integral of $\int_C \mathbf{F}\cdot d\mathbf{r}$, with $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}$ from point A(0, 0, 0) to point B(2, 2, 2) along C: the straight line $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$.

Solution: $\because\ \nabla\times\mathbf{F} = 0$ $\therefore$ the line integral is independent of the path, and the potential is

$$\phi = \int_x F_1 dx = x^2 + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial f}{\partial y} \Rightarrow 2y = \frac{\partial f}{\partial y} \Rightarrow \int_y 2y\,dy = y^2 + g(z)$$

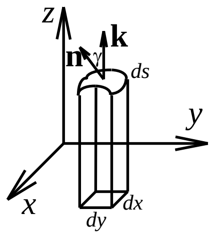
$$\frac{\partial\phi}{\partial z} = \frac{dg}{dz} \Rightarrow 4z = \frac{dg}{dz} \Rightarrow 2z^2 = g(z)$$

$$\phi = x^2 + y^2 + 2z^2$$

and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^B d\phi = \phi(2,2,2) - \phi(0,0,0) = 16$$

II. Surface integrals



Definition: the flux of **F** through the surface S to be the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{s} \quad \text{or} \quad \iint_S \mathbf{F}\cdot\mathbf{n}\,ds,$$

where **n** is the unit normal vector to S.

From the figure shown, $ds\,|\cos\gamma| = dxdy$, γ is the angle between the normal of ds and z-axis.

$$\mathbf{n}\cdot\mathbf{k} = \cos\gamma$$

$$ds = \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

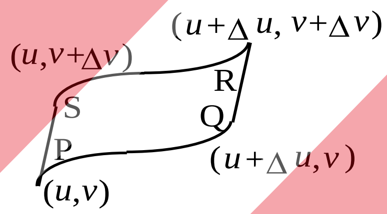
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

Similarly, if α and β are the angle between the normal of ds and x and y axes respectively,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dydz}{|\mathbf{n}\cdot\mathbf{i}|} = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dzdx}{|\mathbf{n}\cdot\mathbf{j}|}$$

When the surface S is parametrized by $\mathbf{r}(u,v)$

$$\overrightarrow{PQ} = \frac{\partial\mathbf{r}}{\partial u}du,\ \overrightarrow{PS} = \frac{\partial\mathbf{r}}{\partial v}dv$$



$$d\mathbf{s} = \overrightarrow{PQ}\times\overrightarrow{PS} = \frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}dudv$$

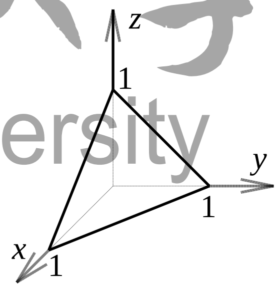
and

$$\iint_S \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}dudv$$

If $\mathbf{F} = \mathbf{n}$, the surface integral is area of the surface itself.

Ex. 4.

Given $\mathbf{F} = x^2\mathbf{i} + 3y^2\mathbf{k}$ and S is the portion of the plane $x + y + z = 1$ in the first octant, evaluate $\iint_S \mathbf{F}\cdot d\mathbf{s}$.



Solution: Let $f = x + y + z - 1$ the unit normal vector **n** of S is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_S \frac{x^2+3y^2}{\sqrt{3}}\frac{dxdy}{1/\sqrt{3}}$$

$$= \iint_S (x^2+3y^2)dxdy = \int_0^1\int_0^{1-x}(x^2+3y^2)dydx$$

$$= \int_0^1 [x^2(1-x)+(1-x)^3]dx = \frac{1}{3}$$

Ex. 5.

Calculate the surface integral of the vector function $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$ over the portion of the surface of the unit sphere S: $x^2+y^2+z^2=1$ above the xy-plane, $z\geq 0$.

Solution: (1)Let $f=x^2+y^2+z^2$, the unit normal vector of S is

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S (x^2+y^2)\frac{dxdy}{z}=4\int_0^1\int_0^{\sqrt{1-x^2}}\frac{x^2+y^2}{\sqrt{1-(x^2+y^2)}}dydx$$

Let $x=r\cos\theta$, $y=r\sin\theta$, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=4\int_0^{\pi/2}\int_0^1\frac{r^2}{\sqrt{1-r^2}}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta=4\int_0^{\pi/2}\int_0^1\frac{r^2}{\sqrt{1-r^2}}r\,drd\theta$$

$$=4\int_0^{\pi/2}\int_0^{\pi/2}\frac{\sin^2\phi}{\cos\phi}\cos\phi\sin\phi\,d\phi d\theta=4\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi d\theta$$

$$=4\int_0^{\pi/2}\left(-\cos\phi+\frac{\cos^3\phi}{3}\right)\Big|_0^{\pi/2}d\theta=4\int_0^{\pi/2}(1-\frac{1}{3})\,d\theta=\frac{4\pi}{3}$$

(2)Let $x=\sin\theta\cos\phi$, $y=\sin\theta\sin\phi$, $z=\cos\theta$, then $\mathbf{r}=\sin\theta\cos\phi\mathbf{i}+\sin\theta\sin\phi\mathbf{j}+\cos\theta\mathbf{k}$,

$\mathbf{F}=\sin\theta\cos\phi\mathbf{i}+\sin\theta\sin\phi\mathbf{j}$, and

$$\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\phi}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0\end{vmatrix}$$

$$=\sin^2\theta\cos\phi\mathbf{i}+\sin^2\theta\sin\phi\mathbf{j}+\sin\theta\cos\theta\mathbf{k}$$

$$\iint_S \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial\phi}d\theta d\phi=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(\sin\theta\cos\phi\mathbf{i}+\sin\theta\sin\phi\mathbf{j})\cdot$$

$$(\sin^2\theta\cos\phi\mathbf{i}+\sin^2\theta\sin\phi\mathbf{j}+\sin\theta\cos\theta\kappa)d\theta d\phi$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta d\phi=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\theta(1-\cos^2\theta)d\theta d\phi$$

$$=\int_0^{2\pi}\left(-\cos\theta+\frac{\cos^3\theta}{3}\right)\Big|_0^{\frac{\pi}{2}}d\phi=\int_0^{2\pi}\left(1-\frac{1}{3}\right)d\phi=\frac{2}{3}\phi\Big|_0^{2\pi}=\frac{4\pi}{3}$$

III. Volume integral

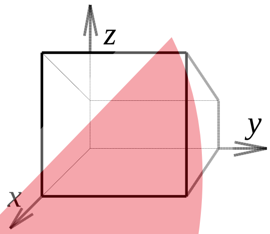
Definition: the volume integral of a function f over the volume V is

$$\iiint_V f\,dv$$

In Cartesian coordinate system, $dv = dxdydz$. If $f = 1$, the volume integral is volume of V.

Ex. 6.

Find the volume integral of $f(x, y, z) = x + yz$ over the box bounded by the coordinate planes, $x = 1$, $y = 2$, and $z = 1 + x$.



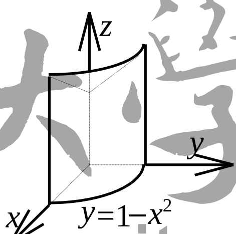
Solution : $\int_0^2\int_0^1\int_0^{1+x}(x+yz)\,dzdxdy = \int_0^2\int_0^1\left(xz+y\frac{z^2}{2}\right)_0^{1+x} dxdy$

$$= \int_0^2\int_0^1\left[x(1+x)+\frac{y}{2}(1+x)^2\right]dxdy = \int_0^2\left[\left(\frac{x^2}{2}+\frac{x^3}{3}\right)+\frac{y}{2}\frac{(1+x)^3}{3}\right]_0^1 dy$$

$$= \int_0^2\left(\frac{5}{6}+\frac{7}{6}y\right)dy = \frac{5}{6}y+\frac{7}{12}y^2\Big|_0^2 = 4$$

Ex. 7.

Find the volume of the region of space above the xy plane and beneath the plane $z = 2 + x + y$, bounded by the planes $y = 0$, $x = 0$, and the surface $y = 1 - x^2$.



Solution : $\int_0^1\int_0^{1-x^2}\int_0^{2+x+y} dzdydx = \int_0^1\int_0^{1-x^2}(2+x+y)\,dydx$

$$= \int_0^1\left[(2+x)y+\frac{y^2}{2}\right]_0^{1-x^2} dx = \int_0^1\left[(2+x)(1-x^2)+\frac{(1-x^2)^2}{2}\right]dx$$

$$= \int_0^1\left(\frac{x^4}{2}-x^3-3x^2+x+\frac{5}{2}\right)dx = \left(\frac{x^5}{10}-\frac{x^4}{4}-x^3+\frac{x^2}{2}+\frac{5}{2}x\right)_0^1$$

$$= \frac{1}{10}-\frac{1}{4}-1+\frac{1}{2}+\frac{5}{2} = \frac{37}{20}$$

IV. Divergence theorem (Gauss theorem)

Divergence theorem: Let V be a closed bounded region in space whose boundary is a piecewise smooth oriented surface S. Let $\mathbf{F}$ be a vector function that is continuous first partial derivative in some domain containing V. Then

$$\oiint_S \mathbf{F}\cdot d\mathbf{s} = \iiint_V \nabla\cdot\mathbf{F}dv.$$

Ex. 8.

Evaluate $\iint_S [xy\mathbf{i} + xz\mathbf{j} + (1 - z - yz)\mathbf{k}]\cdot d\mathbf{s}$, with $S: z = 1 - x^2 - y^2,\ z \geq 0.$

Solution: (1) Let $f = x^2 + y^2 + z$

$$\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}ds = \iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_S (2x^2y + 2xyz + 1 - z - yz)dxdy$$

$$= \iint_S [xy\mathbf{i} + xz\mathbf{j} + (1 - z - yz)\mathbf{k}]\cdot\frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}\frac{dxdy}{\left|\frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}\cdot\mathbf{k}\right|}$$

$$= \iint_S (2x^2y + 2xyz + 1 - z - yz)dxdy$$

$$= \iint_S [2x^2y + 2xy(1 - x^2 - y^2) + 1 - (1 - x^2 - y^2) - y(1 - x^2 - y^2)]dxdy$$

$$= \iint_S [3x^2y + 2xy(1 - x^2 - y^2) + (x^2 + y^2) - y + y^3]dxdy$$

$$= \int_0^{2\pi}\int_0^1 [3r^3\cos^2\theta\sin\theta + 2r^2\sin\theta\cos\theta(1 - r^2) + r^2 - r\sin\theta + r^3\sin^3\theta]rdrd\theta$$

$$= \int_0^{2\pi}\left(\frac{3}{5}\cos^2\theta\sin\theta + \frac{1}{6}\sin\theta\cos\theta + \frac{1}{4} - \frac{1}{3}\sin\theta + \frac{1}{5}\sin^3\theta\right)d\theta$$

$$= \left(-\frac{1}{5}\cos^3\theta - \frac{1}{24}\cos 2\theta + \frac{1}{4}\theta + \frac{1}{3}\cos\theta - \frac{1}{5}\cos\theta + \frac{1}{15}\cos^3\theta\right)\Big|_0^{2\pi} = \frac{\pi}{2}$$

(2) $$\iint_S \mathbf{F}\cdot\mathbf{n}ds = \iiint_V \nabla\cdot\mathbf{F}dv - \iint_{S_1}\mathbf{F}\cdot\mathbf{n}ds = \iiint_V (-1)dv - \iint_{S_1}\mathbf{F}\cdot(-\mathbf{k})dxdy$$

$$= -\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-x^2-y^2} dzdxdy + \iint_{S_1}(1 - z - yz)\big|_{z=0}dxdy$$

$$= -\int_0^{2\pi}\int_0^1 (1 - r^2)rdrd\theta + \pi = -\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4}\right)d\theta + \pi = \frac{\pi}{2}$$

[Exercises]

1. Evaluate $\iint_S [x^2\mathrm{i}-(1+2x)\mathrm{j}+z\mathrm{k}]\cdot ds$, S is the lateral surface of the portion of the cylinder $x^2 + y^2 = 1$ for which $0 \le z \le 1$.

[Solution] (1) Let $f = x^2+y^2-1$, then $\mathbf{n} = x\mathbf{i}+y\mathbf{j}$

$$\iint_S \mathrm{F}\cdot\mathrm{n}\frac{dydz}{|\mathrm{n}\cdot\mathrm{i}|} = \int_0^1\int_{-1}^1 [x^3-y(1+2x)]\frac{dydz}{|x|}$$

$$= \int_0^1\int_{-1}^1 [x^3-y(1+2x)]\frac{dydz}{x} + \int_0^1\int_{-1}^1 [x^3-y(1+2x)]\frac{dydz}{-x} = 0$$

(2) $\iiint_V \nabla\cdot\mathrm{F}dv - \iint_{S_1}\mathrm{F}\cdot\mathrm{k}dxdy - \iint_{S_2}\mathrm{F}\cdot(-\mathrm{k})dxdy$

$$= \int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^1 (2x+1)dzdxdy - \iint_{S_1} 1dxdy + 0$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta+1)rdrd\theta - \pi = \int_0^{2\pi}(\frac{2}{3}\cos\theta+\frac{1}{2})d\theta - \pi = \pi - \pi = 0$$

2. Evaluate $\iint_S (7x\mathrm{i}-z\mathrm{k})\cdot ds$, with S: $x^2+y^2+z^2=4$.

[Solution] (1) $\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/2$

$$\iint_S (7x\mathbf{i}-z\mathbf{k})\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_S \frac{7x^2-z^2}{z}dxdy$$

$$= \iint_S \frac{7x^2-(4-x^2-y^2)}{\sqrt{4-(x^2+y^2)}}dxdy = \iint_S \frac{8x^2+y^2-4}{\sqrt{4-(x^2+y^2)}}dxdy$$

$$= 8\int_0^{\pi/2}\int_0^2 \frac{8r^2-7r^2\sin^2\theta-4}{\sqrt{4-r^2}}rdrd\theta$$

$$= 8\int_0^{\pi/2}(\int_0^2\frac{8r^2-7r^2\sin^2\theta}{\sqrt{4-r^2}}rdr - \int_0^2\frac{4}{\sqrt{4-r^2}}rdr)d\theta$$

$$= 8\int_0^{\pi/2}\left[\int_0^{\pi/2}\frac{8(2\sin\phi)^3-7(2\sin\phi)^3\sin^2\theta}{2\cos\phi}(2\cos\phi\, d\phi) + 2\sqrt{4-r^2}\Big|_0^2\right]d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}(64\sin^3\phi - 56\sin^3\phi\sin^2\theta)\,d\phi\,d\theta - 8\int_0^{\pi/2}8d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2}[(64(1-\cos^2\phi)-56(1-\cos^2\phi)\sin^2\theta](\sin\phi d\phi)\,d\theta - 32\pi$$

$$= 8\int_0^{\pi/2}(128/3 - 112/3\sin^2\theta)\,d\theta - 32\pi = 64\pi$$

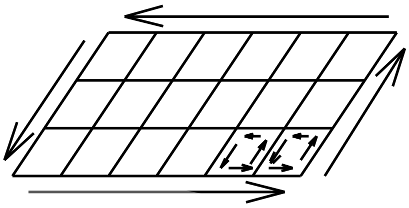
(2) $\nabla\cdot(7x\mathbf{i}-z\mathbf{k}) = 7-1 = 6$

$$\iint_S (7x\mathbf{i}-z\mathbf{k})\cdot d\mathbf{s} = \iiint_V 6dv = 6[\frac{4}{3}\pi(2)^3] = 64\pi$$

V. Stokes theorem

Stokes theorem: Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let **F** be a continuous vector function that has continuous first partial derivative in a domain in space containing S. Then

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}=\oint_C \mathbf{F}\cdot d\mathbf{r}$$



[Note] The positive direction along C is defined as the direction along which an observer, traveling on the positive side of S, would proceed in keeping the enclosed area to his left.

Ex. 9.

Find the surface integral $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}$, with $\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ and S: the paraboloid $z=1-(x^2+y^2)$, $z\ge 0$.

Solution : (1) $\mathbf{n}=\dfrac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4x^2+4y^2+1}}$, $\nabla\times\mathbf{F}=-\mathbf{i}-\mathbf{j}-\mathbf{k}$

$$(\nabla\times\mathbf{F})\cdot\mathbf{n}=\frac{-2x-2y-1}{\sqrt{4x^2+4y^2+1}},\quad \mathbf{n}\cdot\mathbf{k}=\frac{1}{\sqrt{4x^2+4y^2+1}}$$

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(-2x-2y-1)\,dxdy=\int_0^{2\pi}\int_0^1(-2r\cos\theta-2r\sin\theta-1)\,rdrd\theta$$

$$=\int_0^{2\pi}\left(-\frac{2}{3}\cos\theta-\frac{2}{3}\sin\theta-\frac{1}{2}\right)d\theta=-\pi$$

(2) $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{s}=\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $C: x^2+y^2=1$

Let $x=\cos\theta$, $y=\sin\theta$

$$\mathbf{F}\cdot d\mathbf{r}=(\sin\theta\mathbf{i}+0\mathbf{j}+\cos\theta\mathbf{k})\cdot(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})d\theta=-\sin^2\theta\, d\theta$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}-\sin^2\theta\, d\theta=\int_0^{2\pi}-\frac{1-\cos 2\theta}{2}d\theta=-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\Big|_0^{2\pi}=-\pi$$

Ex. 10.

Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the circle $x^2+y^2=4$, $z=-3$, oriented counterclockwise as seen by a person standing at the origin, and $\mathbf{F}=y\mathbf{i}+xz^3\mathbf{j}-zy^3\mathbf{k}$.

Solution:

(1) Let $x=2\cos\theta$, $y=2\sin\theta$

$$\mathbf{F}\cdot d\mathbf{r}=(2\sin\theta\mathbf{i}-54\cos\theta\mathbf{j}+24\sin^3\theta\mathbf{k})\cdot(-2\sin\theta\mathbf{i}+2\cos\theta\mathbf{j})d\theta$$
$$=(-4\sin^2\theta-108\cos^2\theta)d\theta$$
$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(-4\sin^2\theta-108\cos^2\theta)\,d\theta=-112\pi$$

(2) As surface S bounded by C we can take the plane circular disk $x^2+y^2\le 4$ in the plane $z=-3$.

Then $\mathbf{n}=\mathbf{k}$, and $\nabla\times\mathbf{F}|_{z=-3}=(9y^2-27x)\mathbf{i}-28\mathbf{k}$, $(\nabla\times\mathbf{F})\cdot\mathbf{n}=-28$

$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\iint_S -28\,ds=-28(\pi\cdot 2^2)=-112\pi$$

VI. Green's theorem

If $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}$ is a vector function that is continuously differentiable in a domain in the *xy*-plane containing a simply connected bounded close region S whose boundary C is a piecewise smooth simple closed curve. Then the Stokes theorem can be reduced to

$$(\nabla\times\mathbf{F})\cdot\mathbf{n}=(\nabla\times\mathbf{F})\cdot\mathbf{k}=\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}$$
$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{s}=\iint_S \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}dx\,dy$$

and

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C F_1dx+F_2dy$$

hence

$$\iint_S \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}dx\,dy=\oint_C F_1dx+F_2dy$$

which is the Green's theorem.

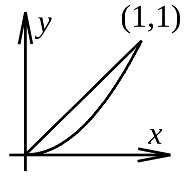
Ex. 11.

Evaluate $\oint_C [(xy+y^2)dx + x^2dy]$, C is a closed curve from (0, 0) to (1, 1) along $y=x^2$ and back to (0, 0) along $y=x$.

Solution:

(1) $\int_0^1 [x\cdot x^2 + (x^2)^2]dx + x^2(2xdx) + \int_1^0 (x\cdot x + x^2)dx + x^2dx = -\frac{1}{20}$

(2) $\iint_S [2x-(x+2y)]dxdy = \int_0^1\int_y^{\sqrt{y}} (x-2y)dxdy = -\frac{1}{20}$



Ex. 12.

Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ counterclockwise around the boundary C of the region R, where $\mathbf{F} = x\sin y\,\mathbf{i} - y\sin x\,\mathbf{j}$, R is the rectangle: $0 \le x \le \pi$, $0 \le y \le \pi/2$.

Solution:

(1) On C_1: $y=0$, $dy=0$, $\mathbf{F}=0 \Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$;

On C_2: $x=\pi$, $dx=0$, $\mathbf{F}=\pi\sin y\,\mathbf{i} \Rightarrow \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$;

On C_3: $y=\pi/2$, $dy=0$, $\mathbf{F} = x\mathbf{i} - \frac{\pi}{2}\sin x\,\mathbf{j} \Rightarrow \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_\pi^0 xdx = -\frac{\pi^2}{2}$;

On C_4: $x=0$, $dx=0$, $\mathbf{F}=0 \Rightarrow \int_{C_4} \mathbf{F}\cdot d\mathbf{r} = 0$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} + \int_{C_4} \mathbf{F}\cdot d\mathbf{r} = -\frac{\pi^2}{2}.$$

(2) $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy = \int_0^{\pi/2}\int_0^{\pi} (-y\cos x - x\cos y)dxdy$

$= \int_0^{\pi/2} \left(-y\sin x - \frac{x^2}{2}\cos y\right)\Big|_0^{\pi} dy = \int_0^{\pi/2} -\frac{\pi^2}{2}\cos y\,dy = -\frac{\pi^2}{2}\sin y\Big|_0^{\pi/2} = -\frac{\pi^2}{2}.$